Chapter

6 Plane Geometry-I

Learning Outcomes

After reading this chapter, you will be able to understand

- Introduction to Euclid Geometry
- Lines and Angles
- Triangles
- Similar Triangles
- Quadrilaterals
- Area of Parallelograms and Triangles

Introduction to Euclid Geometry and Lines and Angles

Introduction to Euclid Geometry

The credit for introducing geometrical concepts goes to the distinguished Greek mathematician **'Euclid'** who is known as the **"Father of Geometry"** and the word **'geometry'** comes from the Greek words **'geo'** which means **'Earth'** and **'metreon'** which means **'measure'**.

Euclid's Definitions

(i) A point is that which has no part.
(ii) A line is breadth-less length.
(iii) The ends of a line segment are points.
(iv) A straight line is that which has length only.
(v) A surface is that which has length and breadth only.
(vi) The edges of a surface are lines.
(vii) A plane surface is that which lies evenly with the straight lines on itself.

Some Undefined Terms

There are three basic concepts in geometry, namely, 'point', 'line' and 'plane'. It is not possible to define these three concepts precisely. We can, however, have a good idea of these concept by considering examples given below

Point: A point is represented by a fine dot made by sharp pencil on a sheet of paper. A Point is denoted by capital letter A, B, P, Q, R etc.

Plane: The surface of a smooth wall, the surface of a sheet of paper, or the surface of a smooth black board are close examples of a plane.

Line: If we fold a piece of paper, the crease in the paper represents a geometrical straight line. The edge of ruler, the edge of the top of a table, the meeting place of two walls of a room are close examples of a geometrical straight line.

Some Geometrical Concepts

Axioms: The basic facts that are taken for granted, without proof, are called axioms.
E.g. Halves of equal are equal.

Statements: A sentence that can be judged to be true or false is called a sentence.
E.g. The sum of the angles of a triangle is 180°, is a true statement.

Theorems: A statement that requires a proof is called a theorem. Establishing the truth of a theorem is known as proving the theorem.
E.g. The sum of all angles around a point is 360°.

Corollary: A statement whose truth can easily be decided from a theorem is called its corollary.

Conjecture: In mathematics, a **conjecture** is a conclusion or proposition based on incomplete information, for which no proof has been found. E.g. **Goldbach Conjecture** states that every even integer greater than two is the sum of two prime numbers.

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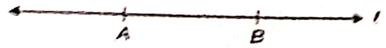
Some Terms Related to Geometry

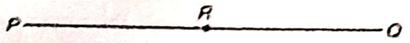
Point: A point is an exact location. A fine dot represents a point. Point is denoted by capital letter *A, B, P, Q, R* etc.

Line segment: The straight path between two points *A* and *B* is called the line segments $\overline{AB}$. The points *A* and *B* are called the end points of the line segment $\overline{AB}$. A line segment has a definite length.

Line: A line segment $\overline{AB}$ when extended indefinitely in both direction is called the line $\overleftrightarrow{AB}$. A line has no end points. A line has no definite length. Sometimes, we label lines by small letters *l, m, n* etc.



Interior point of a line segment: A point *R* is called an interior point of a line segment *PQ* if *R* lies between *P* and *Q* but *R* is neither *P* nor *Q*.

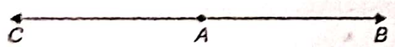


Congruence of line segment: Two line segments *AB* and *CD* are congruent if trace copy of one can be superposed on the other so as to cover it completely and exactly in this case we write $AB \cong CD$. In other words, we can say two lines are congruents if their lengths are same.

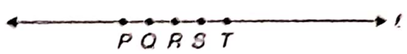
Distance between two points: The distance between two points *P* and *Q* is the length of line segment *PQ*.

Ray: Directed line segment is called a ray. If *AB* is a ray then it is denoted by $\frac{EC}{GC} = \frac{MD}{DC} = \frac{5y/3}{3y} = \frac{5}{9}$. Point *A* is called initial point of ray.

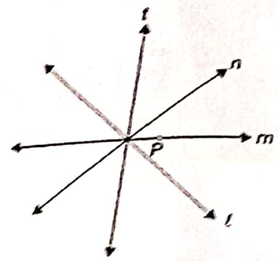
Opposite rays: Two rays *AB* and *AC* are said to be opposite rays if they are collinear and point *A* is the only common point of the two rays.



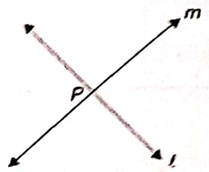
Collinear points: Three or more points are said to be collinear if there is a line which contains all of them.



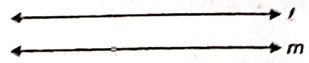
Concurrent Lines: Three or more lines are said to be concurrent if there is a point which lies on all of them.



Intersecting lines: Two lines are intersecting if they have a common point. The common point is called the 'point of intersection'.



Parallel lines: Two lines *l* and *m* in a plane are said to be parallel lines if they do not have a common point.



Plane: A plane is a surface such that every point of the line joining any two points on it lies on it.

Euclid's Axioms

(i) Things which are equal to the same thing are equal to one another.
(ii) If equal are added to equals, the wholes are equal.
(iii) If equal are subtracted from equals, the remainders are equal.
(iv) The things that coincide with one another are equal to one another.
(v) The whole is greater than the part.
(vi) The things that are double of the same thing are equal to one another.
(vii) The thing that are halves of the same thing are equal to one another.

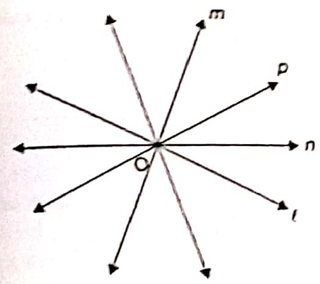
Euclid's Five Postulates

(i) A straight line may be drawn from any one point to any other point.
(ii) A terminated line or a line segment can be produced infinitely.

(iii) A circle can be drawn with any centre and of any radius.
(iv) All right angles are equal to one another.
(v) If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines if produced infinitely meet on that side on which the sum of angles are less than two right angles.

Incidence Axioms

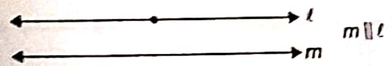
(i) A line is the collection of infinite number of points.
(ii) Through a given point, an infinite line can be drawn.



(iii) Given two distinct points, there is one and only one line that contains both the points.

Parallel Axioms

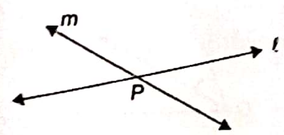
If *P* is a point outside a line ℓ, then one and only one line can be drawn through *P* that is parallel to ℓ.



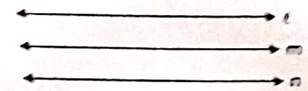
The above axioms may also be re-stated as follows: "Two intersecting lines cannot both be parallel to the same line".

(i) Some more results

(a) Two distinct lines cannot have more than one point in common.



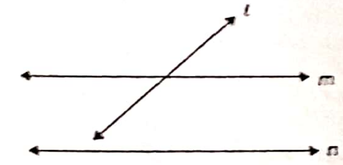
(b) Two lines that are both parallel to the same line, are parallel to each other.
i.e., $\ell \parallel n, m \parallel n \Rightarrow \ell \parallel m$



Theorem: If ℓ, *m*, *n* are lines in the same plane such that ℓ intersects *m* and $n \parallel m$, then ℓ intersects *n* also.

Given: Three lines ℓ, *m*, *n* in the same plane s.t. ℓ intersects *m* and $n \parallel m$.

To prove: Lines l and *n* are intersecting lines.



Proof: Let ℓ and *n* be non-intersecting lines. Then, $\ell \parallel n$.

But, $n \parallel m$ [Given]

$\therefore$ $\ell \parallel n$ and $n \parallel m \Rightarrow \ell \parallel m$

ℓ and *m* are non-intersecting lines.

This is a contradiction to the hypothesis that ℓ and *m* are intersecting lines.

So our supposition is wrong.

Hence, ℓ intersects line *n*.

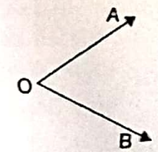
Lines and Angles

Line

A line has length but no width and no thickness.

Angle

An angle is the union of two non–collinear rays with a common initial point. The common initial point is called the **'vertex'** of the angle and two rays are called the **'arms'** of the angles.



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Note

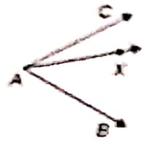
Every angle has a measure and unit of measurement is degree.

One right angle = 90°

1° = 60′ (minutes)

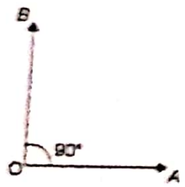
1′ = 60″ (Seconds)

Angle addition axiom: If X is a point in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAX + m\angle XAC$.

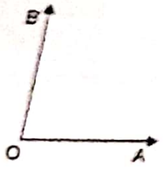


Types of Angles

(i) **Right angle:** An angle whose measure is 90° is called a **right angle**.

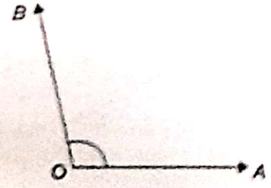


(ii) **Acute angle:** An angle whose measure is less than 90° is called an **acute angle**.



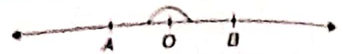
$0° < \angle BOA < 90°$

(iii) **Obtuse angle:** An angle whose measure is more than 90°, but less than 180°, is called an **obtuse angle**.

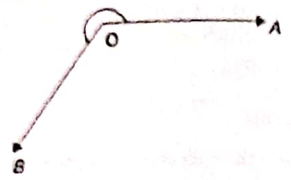


$90° < \angle AOB < 180°$.

(iv) **Straight angle:** An angle whose measure is 180° is called a **straight angle**.

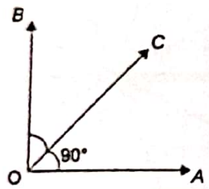


(v) **Reflex angle:** An angle whose measure is more than 180°, but less than 360° is called a **reflex angle**.



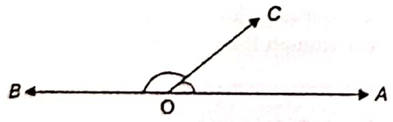
$180° < \angle AOB < 360°$.

(vi) **Complementary angles:** Two angles, the sum of whose measures is 90° are called **complementary angles.**



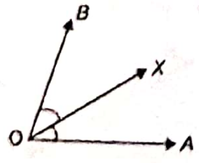
$\angle AOC$ and $\angle BOC$ are complementary as their sum is 90°.

(vii) **Supplementary angles:** Two angles, the sum of whose measures is 180°, are called the **supplementary angles.**



$\angle AOC$ and $\angle BOC$ are supplementary as their sum is 180°.

(viii) **Angle Bisectors:** A ray OX is said to be the bisector of $\angle AOB$, if X is a point in the interior of $\angle AOB$, and $\angle AOX = \angle BOX$.

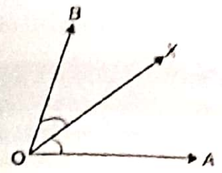


(ix) **Adjacent angles:** Two angles are called **adjacent angles**, if

(A) they have the same vertex,

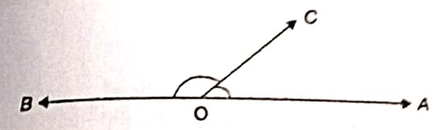
(B) they have a common arm,

(C) non-common arms are on either side of the common arm.



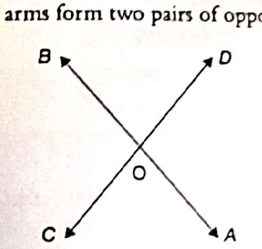
$\angle AOX$ and $\angle BOX$ are adjacent angles, OX is common arm, OA and OB are non-common arms and lie on either side of OX.

(x) **Linear pair of angles:** Two adjacent angles are said to form a linear pair of angles, if their non-common arms are two opposite rays.



$\angle AOC + \angle BOC = 180°$.

(xi) **Vertically opposite angles:** Two angles are called a pair of **vertically opposite angles**, if their arms form two pairs of opposite rays.

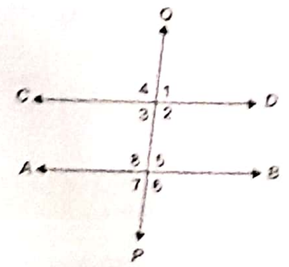


$\angle AOC$ and $\angle BOD$ form a pair of vertically opposite angles. Also $\angle AOD$ and $\angle BOC$ form a pair of vertically opposite angles.

If two lines intersect, then the vertically opposite angles are equal, i.e., $\angle AOC = \angle BOD$ and $\angle BOC = \angle AOD$.

Angles made by a Transversal

Transversal: A line that intersects two or more given lines at distinct points is called a transversal of the given lines.



(i) **Corresponding angles:** Two angles on the same side of a transversal are known as the corresponding angles if both lie either above the two lines or below the two lines, in figure $\angle 1$ and $\angle 5$, $\angle 4$ and $\angle 8$, $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$ are the pairs of corresponding angles.

If a transversal intersects two parallel lines then the corresponding angles are equal i.e., $\angle 1 = \angle 5$, $\angle 4 = \angle 8$, $\angle 2 = \angle 6$ and $\angle 3 = \angle 7$.

(ii) **Alternate interior angles:** $\angle 3$ and $\angle 5$, $\angle 2$ and $\angle 8$, are the pairs of alternate interior angles.

If a transversal intersects two parallel lines then the each pair of alternate interior angles are equal i.e., $\angle 3 = \angle 5$ and $\angle 2 = \angle 8$.

(iii) **Co-interior angles:** The pair of interior angles on the same side of the transversal are called pairs of consecutive or cointerior angles. In figure $\angle 2$ and $\angle 5$, $\angle 3$, and $\angle 8$, are the pairs of co-interior angles.

If a transversal intersects two parallel lines then each pair of consecutive interior angles are supplementary, i.e., $\angle 2 + \angle 5 = 180°$ and $\angle 3 + \angle 8 = 180°$.

Example 1

If a point C lies between two points A and B such that $AC = BC$, then prove that $AC = \frac{1}{2}AB$. Explain by drawing the figure.

Solution

According to the given statement, the figure will be as shown alongside in which the point C lies between two points A and B such that $AC = BC$.

Clearly, $AC + BC = AB$

$\Rightarrow$ $AC + AC = AB$ $\quad [\because AC = BC]$

$\Rightarrow$ $2AC = AB$

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And, $AC = \frac{1}{2} AB$

Example 2

Give a definition for each of the following terms. Are there other terms that need to be defined first? What are they, and how might you define them?

(i) Parallel lines
(ii) Perpendicular lines
(iii) Line segment
(iv) Radius

Solution

(i) Parallel lines: Lines that do not intersect anywhere are called parallel lines.

(ii) Perpendicular lines: Two lines that are at a right angle to each other are called perpendicular lines.

(iii) Line segment: It is a terminated line.

(iv) Radius: The length of the line segment joining the centre of a circle to any point on its circumference is called its radius.

Example 3

How would you rewrite Euclid's fifth postulate so that it would be easier to understand?

Solution

Two distinct intersecting lines cannot be parallel to the same line.

Example 4

Does Euclid's fifth postulate imply the existence of parallel lines? Explain.

Solution

If a straight line ℓ falls on two straight lines m and n such that sum of the interior angles on one side of ℓ is two right angles, then by Euclid's fifth postulate, the line will not meet on this side of ℓ. Next, we know that the sum of the interior angles on the other side of line ℓ also be two right angles. Therefore, they will not meet on the other side. So, the lines m and n never meet and are, therefore, parallel.

Example 5

If lines AB, AC, AD, and AE are parallel to a line ℓ, then prove that points A, B, C, D, and E are collinear.

Solution

Lines AB, AC, AD, and AE are parallel to a line ℓ.

To prove: A, B, C, D, E are collinear.

Proof: Since AB, AC, AD, and AE are all parallel to a line ℓ. Therefore, point A is outside ℓ and lines AB, AC, AD, AE are drawn through A and each line is parallel to ℓ.

But by parallel lines axiom, one and only one line can be drawn through the point A outside it and parallel to ℓ.

This is possible only when A, B, C, D, and E all lie on the same line. Hence, A, B, C, D, and E are collinear.

Example 6

Find the measure of an angle, if six times its complement is 12° less than twice its supplement.

Solution

Let the measure of the required angle be $x°$.

Then, measure of its complement $= (90 - x)°$.

Measure of its supplement $= (180 - x)°$.

$\therefore \quad 6(90 - x) = 2(180 - x) - 12$

$\Rightarrow \quad 540 - 6x = 360 - 2x - 12$

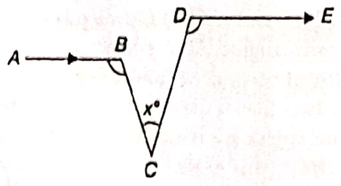
$\Rightarrow \quad 4x = 192$

$\Rightarrow \quad x = 48°$.

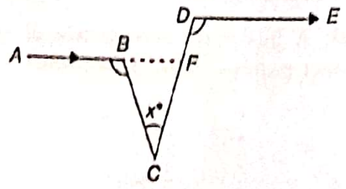
Hence, the required angle is 48°.

Example 7

In the adjoining figure, $\angle ABC = 100°$, $\angle EDC = 120°$, and $AB \parallel DE$. Then, find $\angle BCD$.



Solution



Produce AB to meet CD at E.

$\angle BFD = \angle EDF = 120°$ [alternate interior $\angle$s]

$\Rightarrow \quad \angle BFC = (180° - 120°) = 60°$

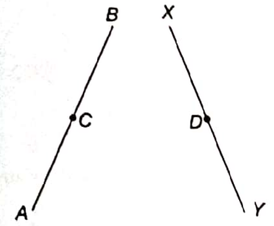
$\Rightarrow \quad \angle CBF = (180° - 100°) = 80°$

$\therefore \quad \angle BCF = 180° - (60° + 80°) = 40°$.

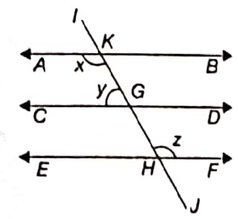
Know Your Progress - 1

1. Solve the equation $a - 15 = 25$ and state which axiom do you use here.

2. In figure, we have $AC = XD$, C is the mid-point of AB and D is the mid-point of XY. Using an Euclid's axiom, show that $AB = XY$.



3. In figure, if $AB \parallel CD \parallel EF$ and $y : z = 3 : 7$, find x.



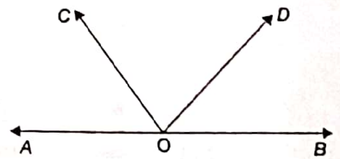
4. Two complementary angles are such that two times the measure of one is equal to three times measure of the other. Find the measure of the large angle.

5. How many lines can pass through
   (i) One point (ii) Two distinct point

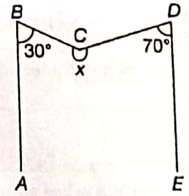
6. Prove that every line segment has one and only one mid-point.

7. Find the supplement of 100° 48′.

8. In figure, find $\angle COD$ when $\angle AOC + \angle BOD = 100°$.

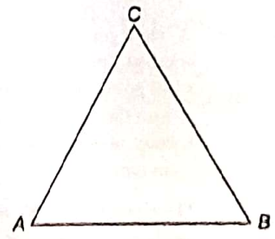


9. In figure, $AB \parallel ED$ and $\angle ABC = 30°$, $\angle EDC = 70°$ then find $x°$.



Triangle

A plane figure bounded by three lines in a plane is called a triangle. Every triangle have three sides and three angles. If ABC is any triangle then AB, BC, and CA are three sides and $\angle A$, $\angle B$, and $\angle C$ are three angles.



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Types of Triangles

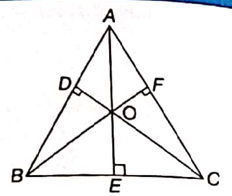
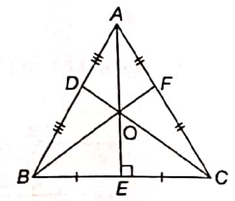
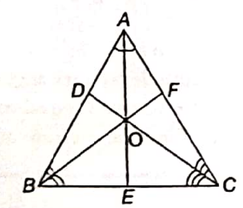
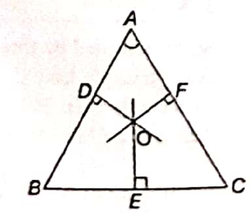
(a) **On the basis of sides, we have three types of triangles.**

1. **Scalene triangle:** A triangle in which no two sides are equal is called a scalene triangle.
2. **Isosceles triangle:** A triangle having two sides equal is called an isosceles triangle.
3. **Equilateral triangle:** A triangle in which all sides are equal is called an equilateral triangle.

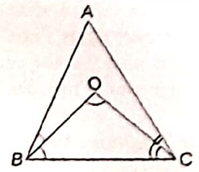
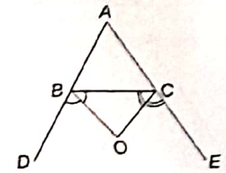
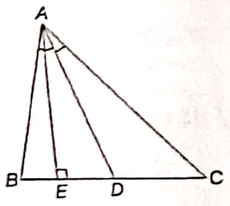
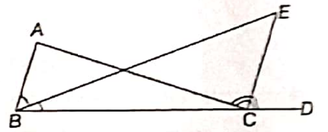
(b) **On the basis of angles, we have three types:**

1. **Right triangle:** A triangle in which any one angle is right angle is called right triangle.
2. **Acute triangle:** A triangle in which all angles are acute is called an acute triangle.
3. **Obtuse triangle:** A triangle in which any one angle is obtuse is called an obtuse triangle.

Some Important Terms

Nomenclature	Property / Definition	Diagram
Altitude (or height)	The perpendicular drawn from the opposite vertex of a side in a triangle called an altitude of the triangle. • There are three altitudes in a triangle. • Point of intersection of the Altitude of the triangle is called as the orthocentre. $\angle BOC = 180° - \angle A$ $\angle COA = 180° - \angle B$ $\angle AOB = 180° - \angle C$	 AE, CD, and BF are the altitudes
Median	The line segment joining the mid-point of a side to the vertex opposite to the side is called a median. • There are three medians in a triangle. • A median bisects the area of the triangle i.e, $Ar(\Delta ABE) = Ar(\Delta AEC) = \frac{1}{2} Ar(\Delta ABC)$ etc. • Point of intersection of medians is called Centroid.	 AE, CD, and BF are the medians ($BE = CE$, $AD = BD$, $AF = CF$)
Angle bisector	A line segment that originates from a vertex and bisects the same angle is called an angle bisector $(\angle BAE = \angle CAE = \frac{1}{2} \angle BAC)$ etc. • Point of intersection of angle bisectors is called incentre.	 AE, CD, and BF are the angles' bisectors
Perpendicular bisector	A line segment that bisects a side perpendicularly (i.e., at right angle) is called a perpendicular bisector of a side of triangle. • All points on the perpendicular bisector of a line are equidistant from the ends of the line. • Point of intersection of perpendicular bisectors is called circumcentre.	 DO, EO, and FO are the perpendicular bisectors
Euler line	The Circumcircle (S), the Centroid (G) and orthocentre (H) of a non-equilateral triangle are collinear and $HG = 2\,GS$. The line passing through H, G, S is called the Euler line.	

Some Important Results

S.No.	Statement	Diagram
1.	In a ΔABC, if the bisectors of $\angle B$ and $\angle C$ meet at O then $\angle BOC = 90° + (\angle A)/2$	
2.	In a ΔABC, if sides AB and AC are produced to D and E, respectively, and the bisectors of $\angle DBC$ and $\angle ECB$ intersect at O, then $\angle BOC = 90° - (\angle A)/2$	
3.	In a ΔABC, if AD is the angle bisector of $\angle BAC$ and $AE \perp BC$, $\angle DAE = \frac{1}{2} (\angle ABC - \angle ACB)$	
4.	In a ΔABC, if side BC is produced to D and bisectors of $\angle ABC$ and $\angle ACD$ meet at E, then $\angle BEC = \frac{1}{2} \angle BAC$	

Congruent Figures

The figures are called congruent if they have same shape and same size. In other words, two figures are called congruent if they are having equal length, width, and height.

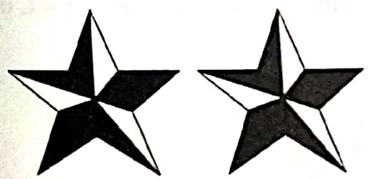
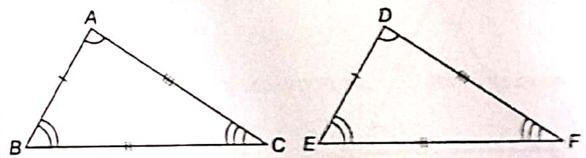


Fig.(i) Fig.(ii)

In the above figures {fig.(i) and fig.(ii)}, both are equal in length, width, and height, so these are congruent figures.

Congruent Triangles

Two triangles are congruent if and only if one of them can be made to superimposed on the other, so as to cover it exactly.



If two triangles ΔABC and ΔDEF are congruent then $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ and $AB = DE$, $BC = EF$, $AC = DF$.

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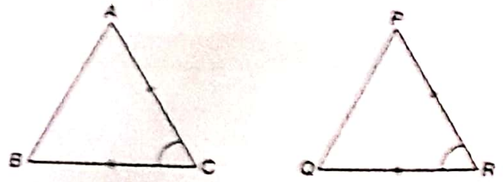
If two ΔABC and ΔDEF are congruent then we write $\Delta ABC \cong \Delta DEF$, we cannot write as $\Delta ABC \cong \Delta DFE$ or $\Delta ABC \cong \Delta EDF$.

Hence, we can say that two triangles are congruent if and only if there exist a one–one correspondence between their vertices such that the corresponding sides and the corresponding angles of the two triangles are equal.

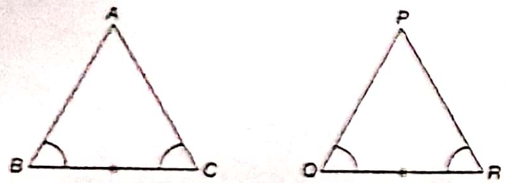
Sufficient Conditions for Congruence of Two Triangles

(i) SAS Congruence Criterion:



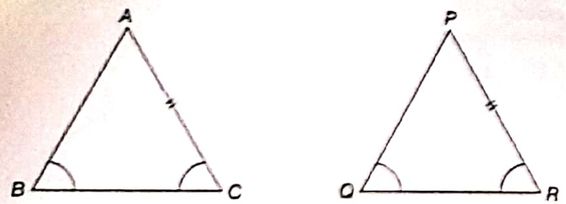
Two triangles are congruent if two sides and the included angle of one are equal to the corresponding sides and the included angle of the other triangle.

(ii) ASA Congruence Criterion:



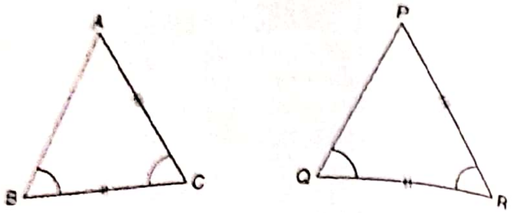
Two triangles are congruent if two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle.

(iii) AAS Congruence Criterion:



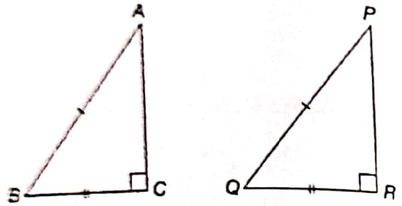
If any two angles and a non-included side of one triangle are equal to the corresponding angles and side of another triangle, then the two triangles are congruent.

(iv) SSS Congruence Criterion:



Two triangles are congruent if the three sides of one triangle are equal to the corresponding three sides of the other triangle.

(v) RHS Congruence Criterion:



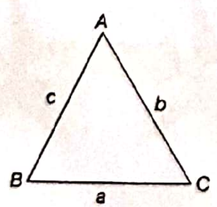
Two right triangles are congruent if the hypotenuse and one side of one triangle are, respectively, equal to the hypotenuse and one side of the other triangle.

Note

If two triangles are congruent, then their corresponding sides and angles are also congruent by *CPCT* (corresponding parts of congruent triangles are also congruent).

Inequality in a Triangle

In triangle ABC, $AB = c$, $BC = a$, and $CA = b$, then,



(i) The sum of any two sides of a triangle is greater than its third side, i.e., in ΔABC,
(A) $a + b > c$ (B) $b + c > a$ (C) $a + c > b$

(ii) If two sides of a triangle are unequal, then the longer side has greater angle opposite to it,
i.e., in ΔABC, if $AB > AC$ then $\angle C > \angle B$.

(iii) Of all the line segments that can be drawn to a given line, from a point, not lying on it, the perpendicular line segment is the shortest

(iv) The difference between any two sides of a triangle is less than its third side, i.e., in ΔABC,
(A) $a - b < c$ (B) $b - c < a$ (C) $a - c < b$

(v) In a right-angled triangle, the sum of squares of two smaller sides is equal to the square of its third side.
i.e., in ΔABC, $a^2 + b^2 = c^2$ (where a and b are smaller as compared to c).

(vi) If sum of squares of two smaller sides is greater than the square of its third side, then that triangle is acute-angled triangle.
i.e., in ΔABC, $a^2 + b^2 > c^2$ (where a and b are smaller as compared to c).

(vii) If sum of squares of two smaller sides is lesser than the square of its third side, then that triangle is obtuse-angled triangle.
i.e., in ΔABC, $a^2 + b^2 < c^2$ (where a and b are smaller as compared to c).

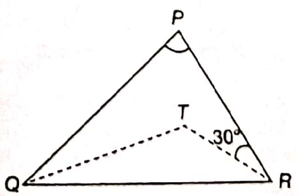
Example 8

In figure, TQ and TR are the bisectors of $\angle Q$ and $\angle R$, respectively. If $\angle QPR = 80°$ and $\angle PRT = 30°$, determine $\angle TQR$ and $\angle QTR$.

Solution

Since the bisectors of $\angle Q$ and $\angle R$ meet at T.

$$\therefore \quad \angle QTR = 90° + \frac{1}{2}\angle QPR$$



$$\Rightarrow \quad \angle QTR = 90° + \frac{1}{2}(80°)$$

$$\Rightarrow \quad \angle QTR = 90° + 40° = 130°$$

In ΔQTR, we have

$$\angle TQR + \angle QTR + \angle TRQ = 180°$$

$$\Rightarrow \quad TQR + 130° + 30° = 180° \qquad [\because \angle TRQ = \angle PRT = 30°]$$

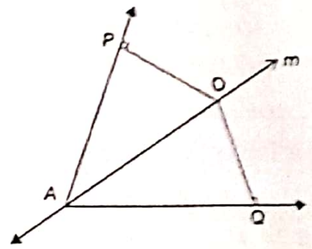
$$\Rightarrow \quad \angle TQR = 20°$$

Thus, $\angle TQR = 20°$ and $\angle QTR = 130°$.

Example 9

In figure, line m is the bisector of an angle $\angle A$ and O is any point on m. OQ and OP are perpendiculars from O to the arms of $\angle A$. Prove that:

(i) $\Delta AOQ \cong \Delta AOP$
(ii) $OQ = OP$.



Solution

(i) In ΔAOQ and ΔAOP,

$\angle OAQ = \angle OAP$ [$\because$ line m is the bisector of $\angle A$]

$\angle AQO = \angle APO = 90°$ [Given]

and $AO = AO$ [Common]

By AAS congruency

$\therefore \quad \Delta AOQ \cong \Delta AOP$

(ii) $OQ = OP$ [C.P.C.T.]

Example 10

If D is the mid–point of the hypotenuse AC of a right triangle ABC, prove that $BD = \frac{1}{2}AC$.

Solution

Given: ΔABC is a right triangle such that $\angle B = 90°$ and D is mid-point of AC.

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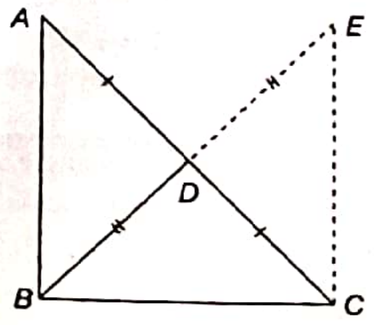
To prove: $BD = \frac{1}{2} AC$.

Construction: Produce BD to E such that $BD = DE$ and join EC.

Proof:



In ΔADB and ΔCDE

$AD = DC$ [Given]

$BD = DE$ [By construction]

And, $\angle ADB = \angle CDE$ [Vertically opposite angles]

$\therefore$ By *SAS* criterion of congruence we have

$$\Delta ADB \cong \Delta CDE$$

$\Rightarrow$ $EC = AB$ and $\angle CED = \angle ABD$ ...(i)

[By CPCT]]

But $\angle CED$ and $\angle ABD$ are alternate interior angles

$\therefore$ $CE \parallel AB$

$\Rightarrow$ $\angle ABC + \angle ECB = 180°$

[Consecutive interior angles]

$90 + \angle ECB = 180° \Rightarrow \angle ECB = 90°$.

Now, In ΔABC and ΔECB, we have

$AB = EC$ [By (i)]

$BC = BC$ [Common]

And, $\angle ABC = \angle ECB = 90°$

$\therefore$ By *SAS* criterion of congruence

$$\Delta ABC \cong \Delta ECB$$

$\Rightarrow$ $AC = EB$ [By CPCT]

$\Rightarrow$ $\frac{1}{2} AC = \frac{1}{2} EB \Rightarrow BD = \frac{1}{2} AC$.

Hence proved.

Example 11

From which given triplet we can make the sides a triangle.

(i) {15, 7, 8}
(ii) {3.5, 4.5, 5.5}

Solution

As we know sum of two sides is always greater than third side.

(i) We cannot make the triangle because here the sum of two sides is equal to third side
i.e., $7 + 8 = 15$

(ii) We can make the triangle because sum of two sides is always greater then third side
$3.5 + 4.5 > 5.5$
$3.5 + 5.5 > 4.5$
$4.5 + 5.5 > 3.5$

Example 12

Prove that any two sides of the triangle are together greater than twice the median drawn to the third side.

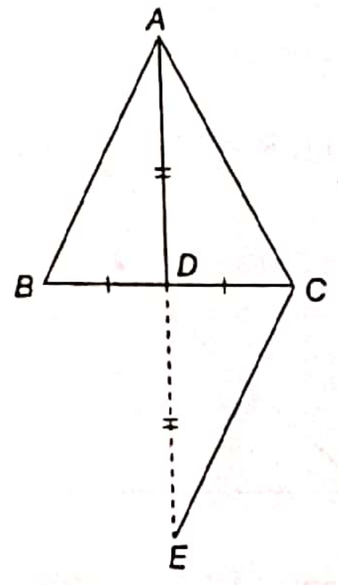
Solution

Given: ΔABC and AD is the median.

To prove: $AB + AC > 2AD$

Construction: Produce AD to E such that $AD = DE$. Join EC.

Proof: In ΔADB and ΔCDE



$AD = DE$ [By construction]

$BD = DC$ [AD is the median]

$\angle ADB = \angle CDE$ [Vertically opposite angles]

$\Delta ADB \cong \Delta CDE$ [By *SAS* congruency]

So, by CPCT

$AB = EC$

In ΔAEC

$AC + EC > 2AD$.

[Sum of two sides of a triangle is always greater than the.....,

So, $AC + AB > 2AD$. [As $EC = AB$].

Example 13

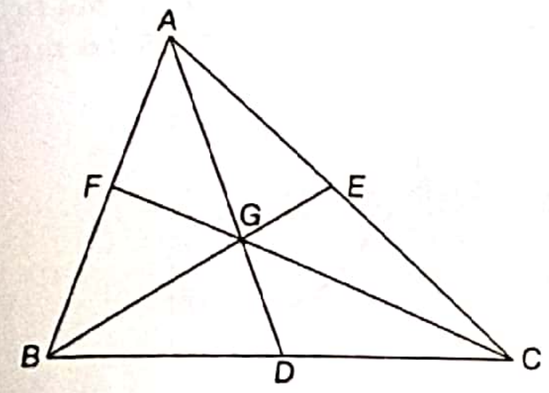
AD, *BE*, and *CF* be medians of a ΔABC.
Prove that $2(AD + BE + CF) < 3(AB + BC + CA) < 4(AD + BE + CF)$.

Solution

In ΔABC, AD, BE, and CF are medians. We know that these are concurrent. Let these meet at G.

Also $AG : GD = BG : GE = CG : GF = 2 : 1$

So, $AG = \frac{2}{3} AD$, $BG = \frac{2}{3} BE$, $CG = \frac{2}{3} CF$.



Because in triangle, the sum of two sides is greater than the third,

So, in ΔAGB, $AG + BG > AB$

in ΔBGC, $BG + CG > BC$

in ΔCGA, $CG + AG > CA$

Adding these, we get

$2(AG + BG + CG) > AB + BC + CA$

or $2\left(\frac{2}{3}AD + \frac{2}{3}BE + \frac{2}{3}CF\right) > AB + BC + CA$

or $\frac{4}{3}(AD + BE + CF) > AB + BC + CA$

or $4(AD + BE + CF) > 3(AB + BC + CA)$ ...(i)

Now, in ΔABD

$AB + BD > AD$

or, $AB + \frac{1}{2} BC > AD$ (Since AD is a median)

Similarly, $BC + \frac{1}{2} CA > BE$ and $CA + \frac{1}{2} AB > CF$

Adding these, we get

$\frac{3}{2}(AB + BC + CA) > AD + BE + CF$

or $3(AB + BC + CA) > 2(AD + BE + CF)$ ...(ii)

Combining (i) and (ii), we get

$2(AD + BE + CF) < 3(AB + BC + CA) < 4(AD + BE + CF)$.

Example 14

In a ΔABC, $AC > AB$. The bisector of $\angle A$ meets BC at E. Prove that $CE > BE$.

Solution

Given: A ΔABC, in which $AC > AB$ and AE bisects $\angle A$

To prove: $CE > BE$.

Construction: Take a point F on AC, such that $AF = AB$. Join BF and EF. Let BF intersect AE at O.

Proof: ΔABF is an isosceles (Since $AB = AF$, by construction)

AE bisects $\angle A$ (given)

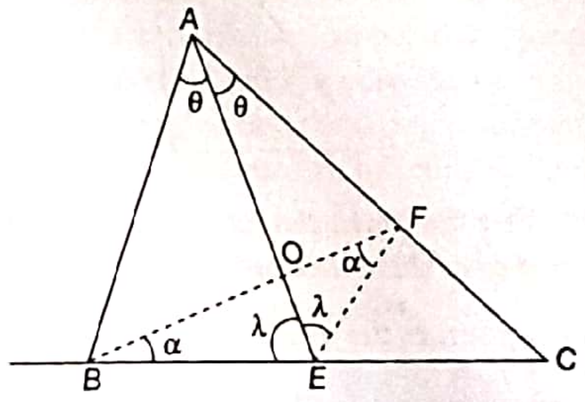
$\therefore$ AE is the perpendicular bisector of BF, so any point on AE is equidistant from B and F. In particular,

$BE = FE$ ...(i)

So, in ΔBEF,

$\angle EBF = \angle EFB = \alpha$, say (angles opposite equal sides)

So, $\angle BEO = 90° - \alpha = \angle FEO$ ($= \lambda$, say)



Thus, AE also bisects $\angle BEF$.

Now, because in a triangle, exterior angle is the sum of interior opposite angles, so

$\angle EFC = \theta + \lambda$ and ext $\angle B = \theta + \lambda$

So, $\angle EFC$ = ext $\angle B$. But $\angle B = 2\theta + \angle C$, so $\angle EFC = 2\theta + \angle C$.

Therefore, $\angle EFC > \angle C$

So, $CE > FE$

Hence $CE > BE$ (using (i)).

Example 15

The perimeter of a triangle is 20. The lengths of three sides are all integers. How many triangle are there are not congruent?

Solution

Method–1:

Let x be the length of the shortest side.

$x = 2$:	(2, 9, 9)	1 triangle
$x = 3$:	(3, 8, 9)	1 triangle
$x = 4$:	(4, 7, 9), (4, 8, 8)	2 triangle
$x = 5$:	(5, 6, 9), (5, 7, 8)	2 triangle
$x = 6$:	(6, 6, 8), (6, 7, 7)	2 triangle

In total, there are 8 such triangles.

Method – 2:

Let x be the length of the longest side.

$x = 9$:	(9, 9, 2), (9, 8, 3), (9, 7, 4), (9, 6, 5),	4 triangle
$x = 8$:	(8, 8, 4), (8, 7, 5), (8, 6, 6),	3 triangle
$x = 7$:	(7, 7, 7)	1 triangle

In total, there are 8 such triangles.

Know Your Progress - 2

1. How many different isosceles triangles have integer side lengths and perimeter of 81 units?
2. In given figure, find the values of x and y, if $QS = RQ$.

3. In the figure, $AB = AC = AD$. Prove that $\angle BCD$ is a right angle.

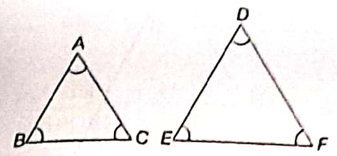
4. Prove that the perimeter of a triangle is greater than the sum of its three medians.
5. O is any point in the interior of a triangle ABC. Prove that:
   (i) $AB + AC > OB + OC$
   (ii) $AB + BC + CA > OA + OB + OC$
   (iii) $OA + OB + OC > \frac{1}{2}(AB + BC + AC)$.
6. In how many ways can square be cut into two congruent triangles?
7. In Figure, ABC is a right triangle and right angled at B such that $\angle BCA = 2\angle BAC$. Show that hypotenuse $AC = 2BC$.

8. Line segment joining the mid-points M and N of parallel sides AB and DC, respectively, of a trapezium $ABCD$ is perpendicular to both the sides AB and DC. Prove that $AD = BC$.
9. ΔABC and ΔDBC are two isosceles triangles on the same base BC, and vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that
   (i) $\Delta ABD \cong \Delta ACD$
   (ii) $\Delta ABP \cong \Delta ACP$
   (iii) AP bisects $\angle A$ as well as $\angle D$
   (iv) AP is the perpendicular bisector of BC.
10. In ΔPQR, S is any point on the side QR. Show that $PQ + QR + RP > 2PS$.

Similar Triangles

Two triangles ABC and DEF are said to be similar if their

(a) Corresponding angles are equal.

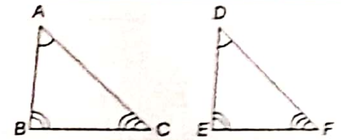
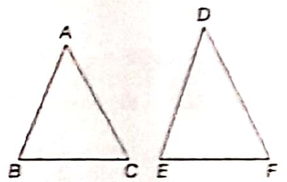
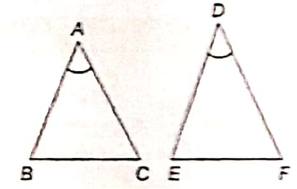


i.e., $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$

(b) Corresponding sides are proportional, i.e.,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

Characteristic Properties of Similar Triangles

Test	Property	Diagram
A – A – A (similarity)	If in two triangles, corresponding angles are equals, i.e., the two triangles are equiangular, then the triangles are similar. $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$ $\Delta ABC \sim \Delta DEF$	
S – S – S (Similarity)	If the corresponding sides of two triangles are proportional, then they are similar. $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$ $\Delta ABC \sim \Delta DEF$	
S – A – S (Similarity)	If in two triangles, one pair of corresponding sides are proportional and the included angles are equal, then the two triangles are similar. $\frac{AB}{DE} = \frac{AC}{DF}$ and $\angle BAC = \angle EDF$ $\Delta ABC \sim \Delta DEF$	

Result Based Upon Characteristic Properties of Similar Triangles

(a) If two triangles are equiangular, then the ratio of the corresponding sides is the same as the ratio of the corresponding medians.

(b) If two triangles are equiangular, then the ratio of the corresponding sides is same as the ratio of the corresponding angle bisector segments.

(c) If two triangles are equiangular, then the ratio of the corresponding sides is same as the ratio of the corresponding altitudes.

(d) If one angle of a triangle is equal to one angle of another triangle and the bisectors of these equal angles divide the opposite side in the same ratio, then the triangles are similar.

(e) If two sides and a median bisecting one of these sides of a triangle are, respectively, proportional to the two sides and the corresponding median of another triangle, then the triangles are similar.

(f) If two sides and a median bisecting the third side of a triangle are, respectively, proportional to the corresponding sides and the median of another triangle, then two triangles are similar.

Note

If the ratio of sides of triangle is $a : b : c$, then ratio of their altitudes is $\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$.

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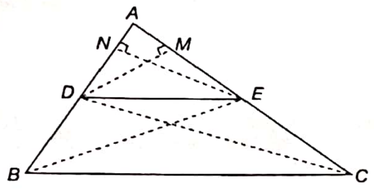
Thales Theorem (Basic Propotionality Theorem)

Statement: If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, then the other two sides are divided in the same ratio.

Given: A triangle ABC in which a line parallel to side BC intersects other two sides AB and AC at D and E, respectively.



To prove: $\frac{AD}{BD} = \frac{AE}{EC}$.

Construction: Join BE and CD and draw $DM \perp AC$ and $EN \perp AB$.

Proof: Area of $\Delta ADE = \frac{1}{2}$ base $\times$ height $= \frac{1}{2} AD \times EN$.

Area of ΔADE is denoted as ar(ADE)

So, $$ar(ADE) = \frac{1}{2} AD \times EN$$

And $$ar(BDE) = \frac{1}{2} DB \times EN,$$

Therefore, $$\frac{ar(ADE)}{ar(BDE)} = \frac{\frac{1}{2} AD \times EN}{\frac{1}{2} DB \times EN} = \frac{AD}{DB} \quad \ldots(i)$$

Similarly, $ar(ADE) = \frac{1}{2} AE \times DM$ and $ar(DEC) = \frac{1}{2} EC \times DM$.

And $$\frac{ar(ADE)}{ar(DEC)} = \frac{\frac{1}{2} AE \times DM}{\frac{1}{2} EC \times DM} = \frac{AE}{EC} \quad \ldots(ii)$$

Note that ΔBDE and ΔDEC are on the same base DE and between the two parallel lines BC and DE.

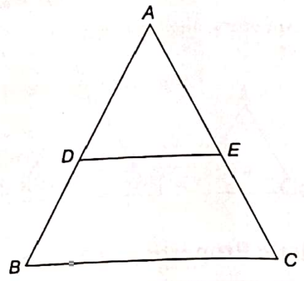
So, $$ar(BDE) = ar(DEC) \quad \ldots(iii)$$

Therefore, from (i), (ii), and (iii), we have:

$$\frac{AD}{DB} = \frac{AE}{EC}$$

Hence proved.

Corollary: If in a ΔABC, a line $DE \parallel BC$, intersects AB in D and AC in E, then



(i) $\frac{DB}{AD} = \frac{EC}{AE}$ (ii) $\frac{AB}{AD} = \frac{AC}{AE}$

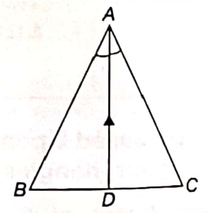
(iii) $\frac{AD}{AB} = \frac{AE}{AC}$ (iv) $\frac{AB}{DB} = \frac{AC}{EC}$

(v) $\frac{DB}{AB} = \frac{EC}{AC}$

Converse of Basic Proportionality Theorem

If a line divides any two sides of a triangle in the same ratio, then the line must be parallel to the third side.

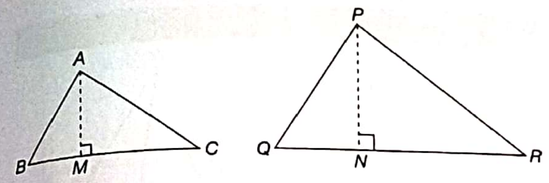
Some Important Results and Theorem

Theorems	Statement/ Explanation	Diagram
Vertical angle bisector	The bisector of the vertical angle of a triangle divides the base in the ratio of other two sides. $\frac{BD}{DC} = \frac{AB}{AC}$	

Areas of Similar Triangle

Theorem: The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

Given: Two triangles ABC and PQR, such that $\Delta ABC \sim \Delta PQR$ [Shown in the figure]



To prove: $\frac{ar(\Delta ABC)}{ar(\Delta PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{CA}{RP}\right)^2$.

Construction: Draw altitudes AM and PN of the triangle ABC and PQR.

Proof: $$ar(ABC) = \frac{1}{2} BC \times AM$$

And $$ar(PQR) = \frac{1}{2} QR \times PN$$

So, $$\frac{ar(ABC)}{ar(PQR)} = \frac{\frac{1}{2} BC \times AM}{\frac{1}{2} QR \times PN} = \frac{BC \times AM}{QR \times PN} \quad \ldots(i)$$

Now, in ΔABM and ΔPQN,

And $\angle B = \angle Q$ [As $\Delta ABC \sim \Delta PQR$]

$\angle M = \angle N$ [90° each]

So, $\Delta ABM \sim \Delta PQN$ [AA similarity criterion]

Therefore, $$\frac{AM}{PN} = \frac{AB}{PQ} \quad \ldots(ii)$$

Also, $\Delta ABC \sim \Delta PQR$ [Given]

So, $$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} \quad \ldots(iii)$$

Therefore, $$\frac{ar(ABC)}{ar(PQR)} = \frac{BC}{QR} \times \frac{AB}{PQ}$$ [From (i) and (ii)]

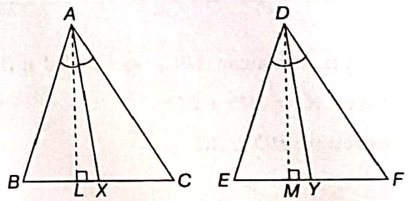
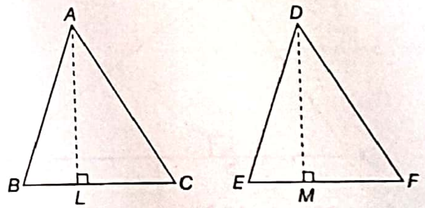
$$= \frac{AB}{PQ} \times \frac{AB}{PQ}$$ [From (iii)]

$$= \left(\frac{AB}{PQ}\right)^2$$

Now using (iii), we get

$$\frac{ar(\Delta ABC)}{ar(\Delta PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{CA}{RP}\right)^2$$

Properties of Areas of Similar Triangle

S. No.	Statement	Diagram
1.	The areas of two similar triangles are proportional to the squares of their corresponding sides. If $\Delta ABC \sim \Delta DEF$, then $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$	
2.	The areas of two similar triangles are proportional to the squares of their corresponding altitude. If $\Delta ABC \sim \Delta DEF$, $AL \perp BC$ and $DM \perp EF$, then $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{AL^2}{DM^2}$	

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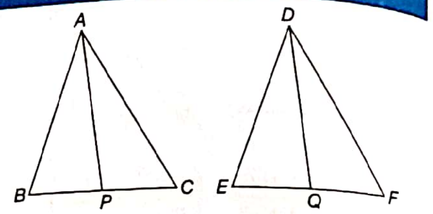
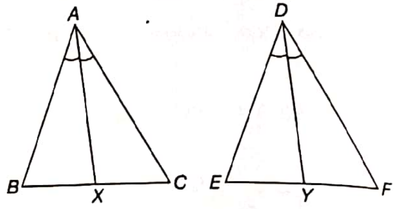
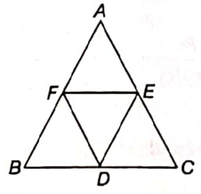
S. No.	Statement	Diagram
3.	The areas of two similar triangles are proportional to the squares of their corresponding medians. If $ABC \sim DEF$ and AP, DQ are their medians, then $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{AP^2}{DQ^2}$	
4.	The areas of two similar triangles are proportional to the squares of their corresponding angle bisector segments. If $\Delta ABC \sim \Delta DEF$ and AX. DY are their bisectors of $\angle A$ and $\angle D$, respectively, then $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{AX^2}{DY^2}$	
5.	If D, E, and F are, respectively, the mid-points of sides BC, CA, and AB of an equilateral triangle ABC then ΔDEF is also an equilateral triangle.	

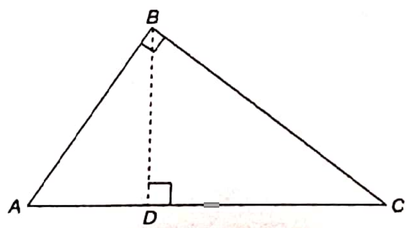
Pythagoras Theorem

Statement: In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Given: A right triangle ABC, right angled at B.

To prove: $AC^2 = AB^2 + BC^2$

Construction: $BD \perp AC$



Proof: ΔADB and ΔABC

$\angle DAB = \angle CAB$ [Common]

$\angle BDA = \angle CBA$ [90° each]

So, $\Delta ADB \sim \Delta ABC$ [By AA similarity]

$$\frac{AD}{AB} = \frac{AB}{AC}$$ [Sides are proportional]

or, $AD \cdot AC = AB^2$ ...(i)

Similarly $\Delta BDC \sim \Delta ABC$

So, $\frac{CD}{BC} = \frac{BC}{AC}$

or $CD \cdot AC = BC^2$ ...(ii)

Adding (i) and (ii),

$AD \cdot AC + CD \cdot AC = AB^2 + BC^2$

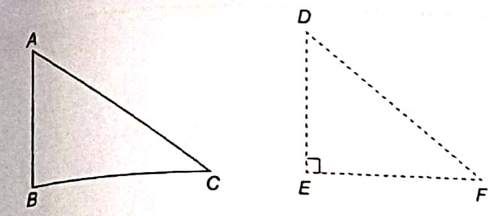
or, $AC(AD + CD) = AB^2 + BC^2$

or, $AC \cdot AC = AB^2 + BC^2$

or, $AC^2 = AB^2 + BC^2$ Hence proved.

Converse of Pythagoras Theorem

Theorem: In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.



Given: A triangle ABC such that $AC^2 = AB^2 + BC^2$

Construction: Construct a triangle DEF such that $DE = AB$, $EF = BC$, and $\angle E = 90°$.

Proof: In order to prove that $\angle B = 90°$, it is sufficient to show $\Delta ABC \sim \Delta DEF$. For this, we proceed as follows.

Since ΔDEF is a right–angled triangle with right angle at E. Therefore, by Pythagoras theorem, we have

$DF^2 = DE^2 + EF^2$

$\Rightarrow$ $DF^2 = AB^2 + BC^2$

[$\because$ $DE = AB$ and $EF = BC$ (By construction)]

$\Rightarrow$ $DF^2 = AC^2$ [$\because$ $AB^2 + BC^2 = AC^2$ (Given)]

$\Rightarrow$ $DF = AC$ ...(i)

Thus, in ΔABC and ΔDEF, we have

$AB = DE$, $BC = EF$ [By construction]

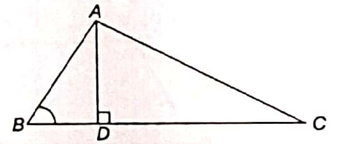
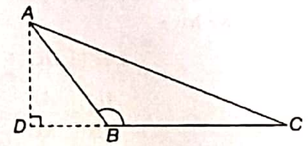
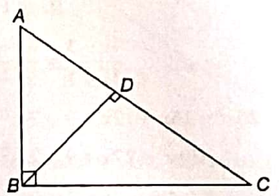
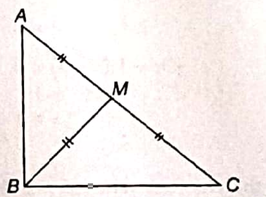
And $AC = DF$ [From equation (i)]

$\therefore$ $\Delta ABC \cong \Delta DEF$ [By SSS criteria of congruency]

$\Rightarrow$ $\angle B = \angle E = 90°$

Hence, ΔABC is a right triangle, right angled at B.

Some Results Deduced From Pythagoras Theorem

S.No.	Statement	Diagram
1.	In an acute angle ΔABC, AD is a perpendicular dropped on the opposite side of $\angle A$, then $AC^2 = AB^2 + BC^2 - 2\ BD \cdot BC$ $(\angle B < 90°)$	
2.	In a obtuse angle ΔABC, AD is perpendicular dropped on $BC \cdot BC$ is produced to D to meet AD, then $AC^2 = AB^2 + BC^2 + 2\ BD\ BC$ $(\angle B > 90°)$	
3.	In a right-angled ΔABC, $\angle B = 90°$ and AC is hypotenuse, the perpendicular BD is dropped on hypotenuse AC from right angle vertex B, then (i) $BD = \frac{AB \times BC}{AC}$ (ii) $AD = \frac{AB^2}{AC}$ (iii) $CD = \frac{BC^2}{AC}$ (iv) $\frac{1}{BD^2} = \frac{1}{AB^2} + \frac{1}{BC^2}$	
4.	In a right-angled triangle, the median to the hypotenuse $= \frac{1}{2} \times$ hypotenuse, i.e., $BM = \frac{AC}{2}$	

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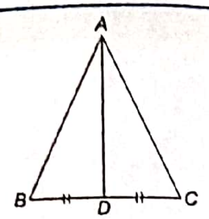
5. Apollonius theorem
In a triangle, the sum of the squares of any two sides of a triangle is equal to twice the sum of the square of the median to the third side and square of half the third side.
i.e., $AB^2 + AC^2 = 2(AD^2 + BD^2)$



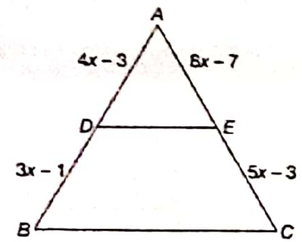
$BD = CD$, AD is the median

Example 16

In a ΔABC, D and E are points on the sides AB and AC, respectively, such that $DE \parallel BC$.

If $AD = 4x - 3$, $AE = 8x - 7$, $BD = 3x - 1$ and $CE = 5x - 3$, find the value of x.

Solution



In ΔABC, we have

$$DE \parallel BC$$

$$\therefore \quad \frac{AD}{DB} = \frac{AE}{EC}$$

[By Basic Proportionality Theorem]

$$\Rightarrow \quad \frac{4x-3}{3x-1} = \frac{8x-7}{5x-3}$$

$$\Rightarrow \quad 20x^2 - 15x - 12x + 9 = 24x^2 - 21x - 8x + 7$$

$$\Rightarrow \quad 20x^2 - 27x + 9 = 24x^2 - 29x + 7$$

$$\Rightarrow \quad 4x^2 - 2x - 2 = 0$$

$$\Rightarrow \quad 2x^2 - x - 1 = 0$$

$$\Rightarrow \quad (2x + 1)(x - 1) = 0$$

$$\Rightarrow \quad x = 1 \text{ or } x = -\frac{1}{2}$$

So, the required value of x is 1 [$x = -\frac{1}{2}$ is neglected as length can not be negative].

Example 17

D and E are, respectively, the points on the sides AB and AC of a ΔABC such that $AB = 12$ cm, $AD = 8$ cm, $AE = 12$ cm, and $AC = 18$ cm, show that $DE \parallel BC$.

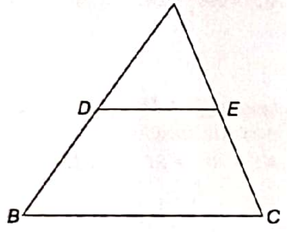
Solution

We have,

$AB = 12$ cm, $AC = 18$ cm, $AD = 8$ cm, and $AE = 12$ cm.

$$\therefore \quad BD = AB - AD = (12 - 8)\text{cm} = 4 \text{ cm}$$

$$CE = AC - AE = (18 - 12)\text{cm} = 6 \text{ cm}$$



Now, $\quad \frac{AD}{BD} = \frac{8}{4} = \frac{2}{1}$

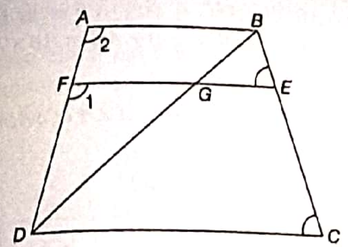
And, $\quad \frac{AE}{CE} = \frac{12}{6} = \frac{2}{1} \Rightarrow \frac{AD}{BD} = \frac{AE}{CE}$

Thus, DE divides sides AB and AC of ΔABC in the same ratio. Therefore, by the converse of basic proportionality theorem, we have $DE \parallel BC$.

Example 18

In a trapezium $ABCD$, $AB \parallel DC$ and $DC = 2AB$. EF drawn parallel to AB cuts AD in F and BC in E such that $\frac{BE}{EC} = \frac{3}{4}$. Diagonal DB intersects EF at G. Prove that $7FE = 10AB$.

Solution



In ΔDFG and ΔDAB,

$$\angle 1 = \angle 2$$

[Corresponding $\angle$s $\because AB \parallel FG$]

$$\angle FDG = \angle ADB \quad \text{[Common]}$$

$$\therefore \quad \Delta DFG \sim \Delta DAB \text{ [By } AA \text{ rule of similarity]}$$

$$\therefore \quad \frac{DF}{DA} = \frac{FG}{AB} \qquad \ldots(i)$$

$$\angle FDG = \angle ADB \quad \text{[Common]}$$

$$\therefore \quad \Delta DFG \sim \Delta DAB \text{ [By } AA \text{ rule of similarity]}$$

$$\therefore \quad \frac{DF}{DA} = \frac{FG}{AB} \qquad \ldots(i)$$

Again in trapezium $ABCD$

$$EF \parallel AB \parallel DC$$

$$\therefore \quad \frac{AF}{DF} = \frac{BE}{EC}$$

$$\Rightarrow \quad \frac{AF}{DF} = \frac{3}{4} \qquad \left[\frac{BE}{EC} = \frac{3}{4} \text{ (given)}\right]$$

$$\Rightarrow \quad \frac{AF}{DF} + 1 = \frac{3}{4} + 1$$

$$\Rightarrow \quad \frac{AF + DF}{DF} = \frac{7}{4}$$

$$\Rightarrow \quad \frac{AD}{DF} = \frac{7}{4} \Rightarrow \frac{DF}{AD} = \frac{4}{7} \qquad \ldots(ii)$$

From (i) and (ii), we get

$$\frac{FG}{AB} = \frac{4}{7} \text{ i.e., } FG = \frac{4}{7} AB \qquad \ldots(iii)$$

In ΔBEG and ΔBCD, we have

$$\angle BEG = \angle BCD$$

[Corresponding angle $\because EG \parallel CD$]

$$\angle GBE = \angle DBC \quad \text{[Common]}$$

$$\therefore \quad \Delta BEG \sim \Delta BCD$$

[By AA rule of similarity]

$$\therefore \quad \frac{BE}{BC} = \frac{EG}{CD}$$

$$\therefore \quad \frac{3}{7} = \frac{EG}{CD}$$

$$\left[\frac{BE}{EG} = \frac{3}{7} \text{ i.e., } \frac{EC}{BE} = \frac{4}{3} \Rightarrow \frac{EC + BE}{BE} = \frac{4+3}{3} \Rightarrow \frac{BC}{BE} = \frac{7}{3}\right]$$

$$\therefore \quad EG = \frac{3}{7} CD = \frac{3}{7} (2AB)$$

[$\because CD = 2AB$ (given)]

$$\therefore \quad EG = \frac{6}{7} AB \qquad \ldots(iv)$$

Adding (iii) and (iv), we get

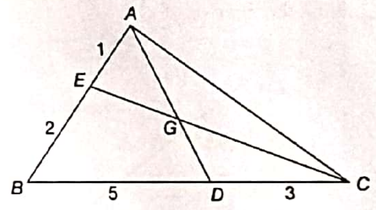
$$FG + EG = \frac{4}{7} AB + \frac{6}{7} AB = \frac{10}{7} AB$$

$$\Rightarrow \quad EF = \frac{10}{7} AB \text{ i.e., } 7EF = 10AB.$$

Hence proved.

Example 19

In the figure $AE : EB = 1 : 2$ and $BD : DC = 5 : 3$. Find $EG : GC$.



Solution

Through E, we draw a line parallel to AD, intersecting BC at M.

$$AE : EB = 1 : 2$$

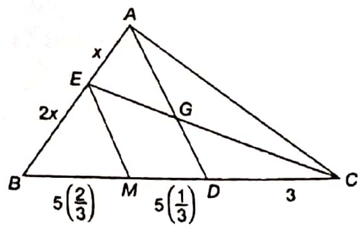
Let $\quad AE = x$, $EB = 2x$,

ΔABD $BD : DC = 5 : 3$ Let $BD = 5y$ $CD = 3y$

$$EM \parallel AD$$

By BDT

Let $Bx = 5x$



$$\frac{BM}{MD} = \frac{BE}{EA} = \frac{2x}{x} = \frac{2}{1}$$

$\therefore$ $$BM = \frac{2}{3} \times BD = \frac{2}{3} \times 5y$$

$$MD = \frac{1}{3} \times 5y$$

$\therefore \Delta CEM$, $GB \parallel EM$

By BPT

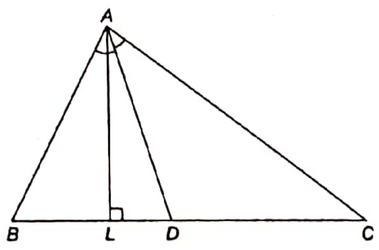
$$\frac{EG}{GC} = \frac{MD}{DC} = \frac{5y/3}{3y} = \frac{5}{9}$$

Example 20

In ΔABC, if AD is the bisector of $\angle A$, prove that $\frac{\text{Area}(\Delta ABD)}{\text{Area}(\Delta ACD)} = \frac{AB}{AC}$.

Solution

In ΔABC, AD is the bisector of $\angle A$.



$\therefore$ $$\frac{AB}{AC} = \frac{BD}{DC} \quad \ldots(i)$$

[By internal bisector theorem]

From A draw $AL \perp BC$

$\therefore$ $$\frac{\text{Area}(\Delta ABD)}{\text{Area}(\Delta ACD)} = \frac{\frac{1}{2} BD \cdot AL}{\frac{1}{2} DC \cdot AL} = \frac{BD}{DC} = \frac{AB}{AC}$$

[From (i)]

Hence proved.

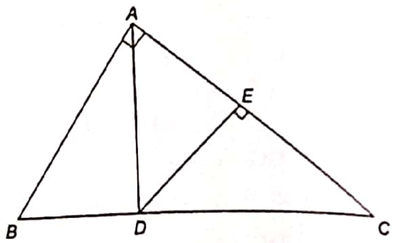
Example 21

In triangle ABC, $\angle BAC = 90°$, AD is its bisector. If $DE \perp AC$, prove that $DE \times (AB + AC) = AB \times AC$.

Solution

It is given that AD is the bisector of $\angle A$ of ΔABC.

$\therefore$ $$\frac{AB}{AC} = \frac{BD}{DC}$$



$\Rightarrow$ $$\frac{AB}{AC} + 1 = \frac{BD}{DC} + 1$$ [Adding 1 on both sides]

$\Rightarrow$ $$\frac{AB + AC}{AC} = \frac{BD + DC}{DC}$$

$\Rightarrow$ $$\frac{AB + AC}{AC} = \frac{BC}{DC} \quad \ldots(i)$$

In Δ's CDE and CBA, we have

$\angle DCE = \angle BCA$ [Common]

$\angle DEC = \angle BAC$ [Each equal to 90°]

So, by AA–criterion of similarity

$\Delta CDE \sim \Delta CBA$

$\Rightarrow$ $$\frac{CD}{CB} = \frac{DE}{BA}$$

$\Rightarrow$ $$\frac{AB}{DE} = \frac{BC}{DC} \quad \ldots(ii)$$

From (i) and (ii), we have

$\Rightarrow$ $$\frac{AB + AC}{AC} = \frac{AB}{DE}$$

$\Rightarrow$ $DE \times (AB + AC) = AB \times AC$.

Example 22

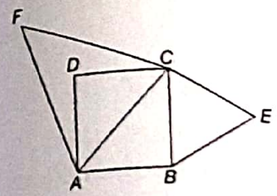
Prove that the area of the equilateral triangle described on the side of a square is half the area of the equilateral triangle described on its diagonal.

Solution

Given: A square $ABCD$. Equilateral triangles ΔBCE and ΔACF have been described on side BC and diagonal AC respectively.

To prove: Area $(\Delta BCE) = \frac{1}{2} \cdot \text{Area}(\Delta ACF)$

Proof: Since ΔBCE and ΔACF are equilateral. Therefore, they are equiangular (each angle being equal to 60°) and hence $\Delta BCE \sim \Delta ACF$.



[$\because$ ABCD is a square
$\therefore$ Diagonal = $\sqrt{2}$(side)
$\Rightarrow$ AC = $\sqrt{2}$BC]

$\Rightarrow$ $$\frac{\text{Area}(\Delta BCE)}{\text{Area}(\Delta ACF)} = \frac{BC^2}{AC^2}$$

$\Rightarrow$ $$\frac{\text{Area}(\Delta BCE)}{\text{Area}(\Delta ACF)} = \frac{BC^2}{\left(\sqrt{2}BC\right)^2} = \frac{1}{2}$$

$\Rightarrow$ $$\frac{\text{Area}(\Delta BCE)}{\text{Area}(\Delta ACF)} = \frac{1}{2}.$$

Hence proved.

Example 23

BL and CM are medians of ΔABC right angled at A. Prove that $4(BL^2 + CM^2) = 5\,BC^2$.

Solution

In ΔBAL

$\Rightarrow$ $$BL^2 = AL^2 + AB^2 \quad \ldots(i)$$

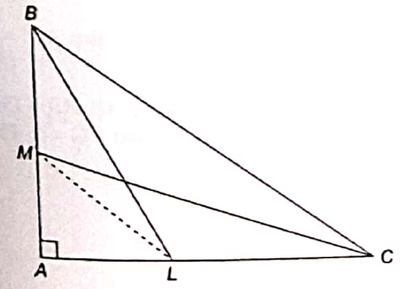
[Using Pythagoras theorem]

and, In ΔCAM

$$CM^2 = AM^2 + AC^2 \quad \ldots(ii)$$

[Using Pythagoras theorem]

Adding (1) and (2) and then multiplying by 4, we get



$$4(BL^2 + CM^2) = 4(AL^2 + AB^2 + AM^2 + AC^2)$$

$$= 4\{AL^2 + AM^2 + (AB^2 + AC^2)\}$$

[$\because \Delta ABC$ is a right triangle]

$$= 4(AL^2 + AM^2 + BC^2) = 4(ML^2 + BC^2)$$

[$\because \Delta LAM$ is a right triangle]

$$= 4ML^2 + 4\,BC^2$$

[A line joining mid–points of two sides is parallel to third side and is equal to half of it, $ML = BC/2$]

$$= BC^2 + 4BC^2 = 5BC^2.$$

Hence proved.

Example 24

In the given figure, $BC \perp AB$, $AE \perp AB$ and $DE \perp AC$. Prove that $DE \cdot BC = AD \cdot AB$.

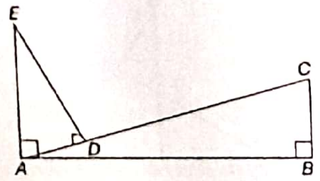
Solution

In ΔABC and ΔEDA,

We have

$\angle ABC = \angle ADE$ [Each equal to 90°]

$\angle ACB = \angle EAD$ [Alternate angles]



$\therefore$ By AA Similarity

$\Delta ABC \sim \Delta EDA$

$\Rightarrow$ $$\frac{BC}{AB} = \frac{AD}{DE}$$

$\Rightarrow$ $DE \cdot BC = AD \cdot AB$.

Hence proved.

Example 25

O is any point inside a rectangle $ABCD$ (shown in the figure). Prove that $OB^2 + OD^2 = OA^2 + OC^2$.

Solution

Through O, draw $PQ \parallel BC$ so that P lies on AB and Q lies on DC.

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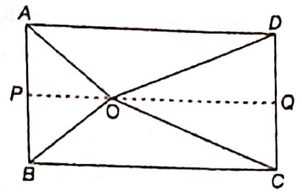


Now, $PQ \parallel BC$

Therefore,

$$PQ \perp AB \text{ and } PQ \perp DC$$

$$[\angle B = 90^\circ \text{ and } \angle C = 90^\circ]$$

So, $\angle BPQ = 90^\circ$ and $\angle CQP = 90^\circ$

Therefore, $BPQC$ and $APQD$ are both rectangles.

Now, from ΔOPB,

$$OB^2 = BP^2 + OP^2 \quad \ldots(i)$$

Similarly, from ΔODQ,

$$OD^2 = OQ^2 + DQ^2 \quad \ldots(ii)$$

From ΔOQC, we have

$$OC^2 = OQ^2 + CQ^2 \quad \ldots(iii)$$

And from ΔOAP, we have

$$OA^2 = AP^2 + OP^2 \quad \ldots(iv)$$

Adding (i) and (ii)

$$OB^2 + OD^2 = BP^2 + OP^2 + OQ^2 + DQ^2$$

$$= CQ^2 + OP^2 + OQ^2 + AP^2$$

[As $BP = CQ$ and $DQ = AP$]

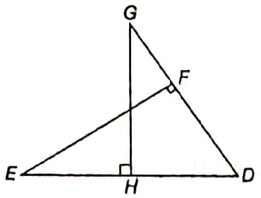
$$= CQ^2 + OQ^2 + OP^2 + AP^2 = OC^2 + OA^2$$

[From (iii) and (iv)]

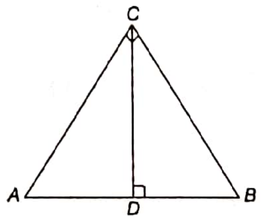
Hence proved.

Know Your Progress - 3

1. In figure, $\angle GHE = \angle DFE = 90^\circ$, $DH = 8$, $DF = 12$, $DG = 3x - 1$ and $DE = 4x + 2$. Find DE.



2. In figure, $\angle ACB = 90^\circ$, $CD \perp AB$, prove that $CD^2 = BD \cdot AD$.



3. In the figure, $PQRS$ is a parallelogram with $PQ = 16$ cm and $QR = 10$ cm. L is a point on PR such that $RL : LP = 2 : 3$. QL produced meets RS at M and PS produced at N.

Find the lengths of PN and RM.

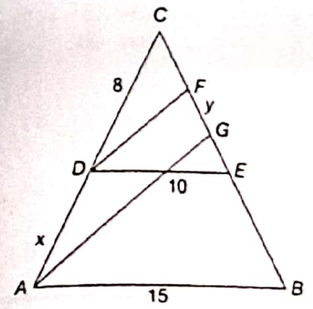
4. ABC is a right triangle, right-angled at C. Let $BC = a$, $CA = b$, $AB = c$, and let p be the length of perpendicular from C on AB. Prove that:

(i) $cp = ab$ (ii) $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$

5. In an equilateral triangle ABC, the side BC is trisected at D. Prove that $9AD^2 = 7AB^2$.

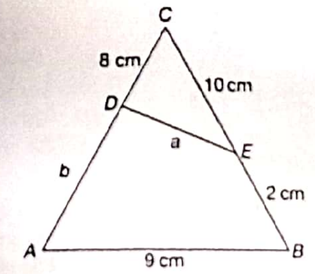
6. In ΔABC, $\angle ABC = 135^\circ$. Prove that $AC^2 = AB^2 + BC^2 + 4ar(\Delta ABC)$.

7. In the figure $DF \parallel AG$, $DE \parallel AB$, $AB = 15$, $CD = 8$, $AD = x$, $DE = 10$, $FG = y$, and $CG = 6$. The ratio $x : y$ equal to:



(A) 1 : 2 (B) 1 : 3
(C) 2 : 1 (D) 3 : 2

8. In the figure $\angle A = \angle CED$, $CD = 8$ cm $CE = 10$ cm, $BE = 2$ cm, $AB = 9$ cm, $AD = b$ and $DE = a$. The value of $a + b$ is:



(A) 13 cm (B) 15 cm
(C) 12 cm (D) 9 cm

9. In ΔABC, D is a point on BC such that $3BD = BC$. If each side of the triangle is 12 cm, then AD equals

(A) $4\sqrt{5}$ (B) $4\sqrt{6}$
(C) $4\sqrt{7}$ (D) $4\sqrt{11}$

10. In ΔABC, $\overline{XY}$ is parallel to $\overline{AC}$ and divides the triangle into the two parts of equal area. Then, $\dfrac{AX}{AB}$ equals

(A) $\dfrac{\sqrt{2}+1}{2}$ (B) $\dfrac{2-\sqrt{2}}{2}$

(C) $\dfrac{2+\sqrt{2}}{2}$ (D) $\dfrac{\sqrt{2}-1}{2}$

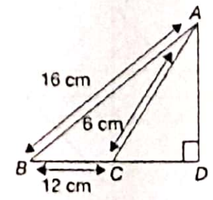
11. $PQRS$ is a smallest square whose vertices are the opposite sides of the square $ABCD$. The ratio of the area of $PQRS$ to $ABCD$ is

(A) 1: 2 (B) 1: $\sqrt{2}$
(C) 1: 3 (D) 2: 3

12. The length of a ladder is exactly equal to the height of the wall it is leaning against. If the lower end of the ladder is kept on a bench of height 3 m and the bench is kept 9 m away from the wall, the upper end of the ladder coincides with the top of the wall. The height of the wall is:

(A) 11 m (B) 12 m
(C) 15 m (D) 18 m

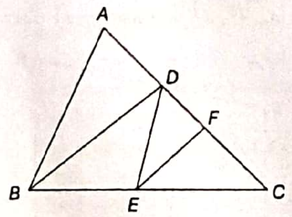
13. In the figure $\angle D = 90^\circ$, $AB = 16$ cm, $BC = 12$ cm, and $CA = 6$ cm, then CD is:



(A) $\dfrac{13}{6}$ cm (B) $\dfrac{17}{6}$ cm

(C) $\dfrac{19}{6}$ cm (D) $\dfrac{18}{5}$ cm

14. In the given figure, ΔABC has points D and F in $\overline{AC}$ and point E in $\overline{BC}$ such that $\overline{DE} \parallel \overline{AB}$ and $\overline{EF} \parallel \overline{BD}$. If $CF = 4$ cm and $AC = 9$ cm, what is the length of $\overline{DC}$?



(A) 7 cm (B) 6 cm
(C) 5 cm (D) 4 cm

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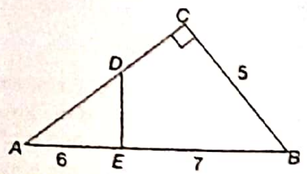
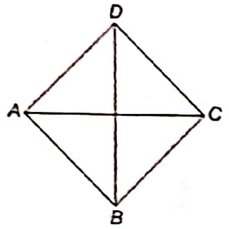
15. In the figure, C is a right angle, $DE \perp AB$, $AE = 6$, $EB = 7$, and $BC = 5$. The area of the quadrilateral EBCD is:



(A) 27.5 (B) 25
(C) 22.5 (D) 20

Quadrilateral

A **quadrilateral** is a four-sided closed figure.

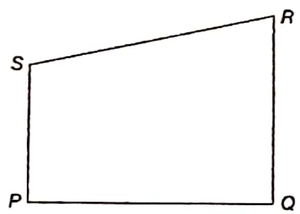


Let A, B, C, and D be four points in a plane such that:

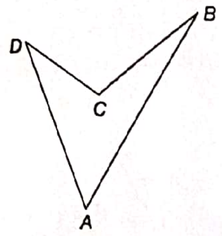
(i) No three of them are collinear.
(ii) The line segments AB, BC, CD, and DA do not intersect except at their end points, then figure obtained by joining A, B, C, and D is called a quadrilateral.

Convex and Concave Quadrilaterals

(i) A quadrilateral in which the measure of each interior angle is less than 180° is called a **convex quadrilateral.**
In figure, $PQRS$ is convex quadrilateral.

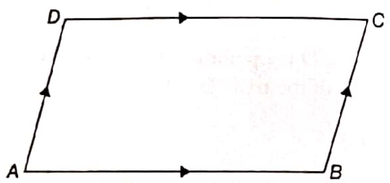


(ii) A quadrilateral in which the measure of one of the interior angles is more than 180° is called a **concave quadrilateral.** In figure, $ABCD$ is concave quadrilateral.



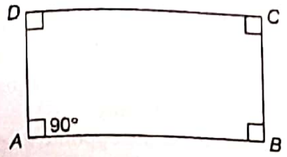
Special Quadrilaterals

(i) **Parallelogram:** A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel. In figure, $AB \parallel DC$, $AD \parallel BC$. Therefore, $ABCD$ is a parallelogram.

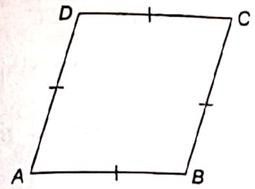


Properties:
(a) A diagonal of a parallelogram divides it into two congruent triangles.
(b) In a parallelogram, opposite sides are equal.
(c) The opposite angles of a parallelogram are equal.
(d) The diagonals of a parallelogram bisect each other.

(ii) **Rectangle:** A rectangle is a parallelogram, in which each of its angle is a right angle. If $ABCD$ is a rectangle, then $\angle A = \angle B = \angle C = \angle D = 90°$, $AB = CD$, $BC = AD$, and diagonals $AC = BD$.

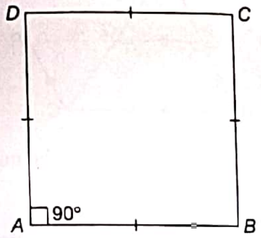


(iii) **Rhombus:** A rhombus is a parallelogram in which all its sides are equal in length. If $ABCD$ is a rhombus, then $AB = BC = CD = DA$.



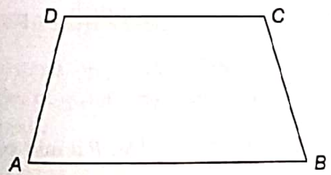
The diagonals of a rhombus are perpendicular to each other.

(iv) **Square:** A square is a parallelogram having all sides equal and each angle equal to right angle. If $ABCD$ is a square, then $AB = BC = CD = DA$, diagonal $AC = BD$, and $\angle A = \angle B = \angle C = \angle D = 90°$.

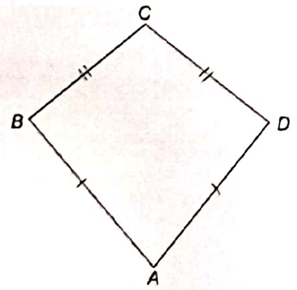


The diagonals of a square are perpendicular to each other.

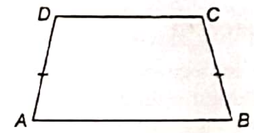
(v) **Trapezium:** A trapezium is a quadrilateral with only one pair of opposite sides parallel. In figure, $ABCD$ is a trapezium with $AB \parallel DC$.



(vi) **Kite:** A kite is a quadrilateral in which two pairs of adjacent sides are equal. If $ABCD$ is a kite then $AB = AD$ and $BC = CD$.



(vii) **Isosceles trapezium:** A trapezium is said to be an isosceles trapezium, if its non–parallel sides are equal. Thus, a quadrilateral $ABCD$ is an isosceles trapezium, if $AB \parallel DC$ and $AD = BC$.



In isosceles trapezium $\angle A = \angle B$ and $\angle C = \angle D$.

Properties

Theorem 1: The sum of the four angles of a quadrilateral is 360°.
Theorem 2: A diagonal of a parallelogram divides it into two congruent triangles.
Theorem 3: In a parallelogram, opposite sides are equal.
Theorem 4: The opposite angles of a parallelogram are equal.
Theorem 5: The diagonals of a parallelogram bisect each other.
Theorem 6: Each of the four angles of a rectangle is a right angle.
Theorem 7: Each of the four sides of a rhombus is of the same length.
Theorem 8: Each of the angles of a square is a right angle and each of the four sides is of the same length.
Theorem 9: The diagonals of a rectangle are of equal length.
Theorem 10: The diagonals of a rhombus are perpendicular to each other.
Theorem 11: The diagonals of a square are equal and perpendicular to each other.

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A quadrilateral become a parallelogram when:

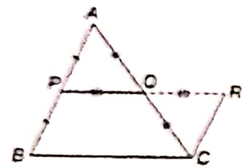
(i) Opposite angles are equal
(ii) Both the pair of opposite sides are equal
(iii) A pair of opposite side is equal as well as parallel
(iv) Diagonals of quadrilateral bisect each other.

Note

(i) Square, rectangle, and rhombus are all parallelograms.
(ii) Kite and trapezium are not parallelograms.
(iii) A square is a rectangle.
(iv) A square is a rhombus.
(v) A parallelogram is a trapezium.

Mid-Point Theorem

In a triangle, the line segment joining the mid-points of any two sides is parallel to the third side and is half of it.



Given: A triangle ABC in which P is the mid–point of side AB and Q is the mid-point of side AC.

To prove: PQ is parallel to BC and is half of it

i.e., $PQ \parallel BC$ and $PQ = \frac{1}{2} BC$.

Construction: Produce PQ upto point R such that $PQ = QR$. Join R and C.

Proof: In ΔAPQ and ΔCRQ

$PQ = QR$ [By construction]
$AQ = QC$ [Given]
And, $\angle AQP = \angle CQR$ [Vertically opposite angles]
So, $\Delta APQ \cong \Delta CRQ$ [By SAS]
$\therefore$ $AP = CR$ [By CPCT]
And, $\angle APQ = \angle CRQ$ [By CPCT]

But, $\angle APQ$ and $\angle CRQ$ are alternate angles and whenever the alternate angles are equal, the lines are parallel.

$\Rightarrow$ $AP \parallel CR \Rightarrow AB \parallel CR \Rightarrow BP \parallel CR$
$AP = BP$ [Given, P is mid–point of AB]
$\Rightarrow$ $CR = BP$ [As, $AP = QP$]

Now, $BP = CR$ and $BP \parallel CR \Rightarrow BCRP$ is a parallelogram [When any pair of opposite sides are equal and parallel the quadrilateral is a parallelogram]

$BCRP$ is a parallelogram and opposite sides of a parallelogram are equal and parallel.

$\therefore$ $PR = BC$ and $PR \parallel BC$

Since,

$$PQ = QR \Rightarrow PQ = \frac{1}{2} PR = \frac{1}{2} BC \quad [\text{As, } PR = BQ]$$

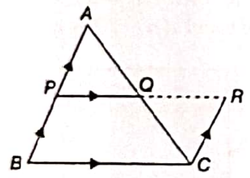
Also, $PQ \parallel BC$ [As, $PR \parallel BQ$]

$\therefore$ $PQ \parallel BC$ and $PQ = \frac{1}{2} BC$

Hence proved.

Converse of the Mid–Point Theorem

The line drawn through the mid–point of one side of a triangle parallel to the another side; bisects the third side.



Given: A triangle ABC in which P is the mid–point of side AB and PQ is parallel to BC.

To prove: PQ bisects the third side AC i.e., $AQ = QC$.

Construction: Through C, draw CR parallel to BA, which meets PQ produced at point R.

Proof: Since, $PQ \parallel BC$ i.e., $PR \parallel BC$ [Given]
and $CR \parallel BA$ i.e., $CR \parallel BP$ [By construction]
$\therefore$ Opposite sides of quadrilateral $PBCR$ are parallel.
$\Rightarrow$ $PBCR$ is a parallelogram
$\Rightarrow$ $BP = CR$
Also, $BP = AP$ [As, P is mid–point of AB]
$\therefore$ $CR = AP$
$\because$ $AB \parallel CR$ and AC is transversal, $\angle PAQ = \angle RCQ$ [Alternate angles]
$\therefore$ $AB \parallel CR$ and PR is transversal, $\angle APQ = \angle CRQ$ [Alternate angles]

In ΔAPQ and ΔCRQ

$CR = AP$, $\angle PAQ = \angle RCQ$ and $\angle APQ = \angle CRQ$
$\Rightarrow$ $\Delta APQ \cong \Delta CRQ$ [By ASA]
$\Rightarrow$ $AQ = QC$
Hence proved.

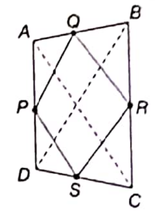
Varignon's Theorem-I

The quadrilateral formed by joining the mid point of consecutive sides of any quadrilateral is a parallelogram.

Given: $ABCD$ is parallelogram. P, Q, R and S are mid points of sides AD, AB, BC and DC respectively.

To prove: $PQRS$ is parallelogram.

Proof: $PQ \parallel BD$
$RS \parallel BD$
$\Rightarrow$ $PQ \parallel RS$



Similarly
$QR \parallel AC$
$PS \parallel AC$
$\Rightarrow$ $QR \parallel PS$
Hence $PQRS$ is parallelogram.

Varignon's Corollary: Rectangle

The quadrilateral formed by joining the mid points of consecutive sides of quadrilateral whose diagonals are perpendicular is a rectangle.

Varignon's Corollary: Rhombus

The quadrilateral formed by joining the mid points of consecutive sides of quadrilateral whose diagonals are congruent is rhombus.

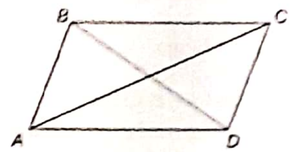
Varignon's Corollary: Square

The quadrilateral formed by joining the mid points of consecutive sides of quadrilateral whose diagonals are congruent and perpendicular is a square.

Some important points

1. The segment joining the mid points of the opposite sides of any quadrilateral bisect each other.
2. The sum of the squares of the lengths of the sides of a parallelogram equals the sum of the squares of the lengths of the diagonals.

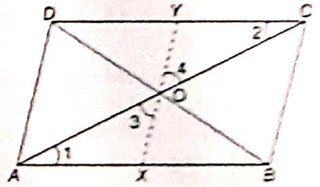
$$AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2$$



3. The angle bisectors of a parallelogram form a rectangle.
4. The angle bisector of a rectangle form a square.

Example 26

The diagonals of a parallelogram $ABCD$ intersect at O. A line through O intersects AB at X and DC at Y. Prove that $OX = OY$.



Solution

Since $AB \parallel CD$
In ΔOAX and ΔOCY
$\angle 1 = \angle 2$ [Alternate angles]
$OA = OC$ and, $\angle 3 = \angle 4$ [Vertically opposite angles]
So, by ASA criterion of congruence,
$\Delta OAX \cong \Delta OCY$
$\Rightarrow$ $OX = OY$.

Example 27

The angle bisectors of a parallelogram form a rectangle.

Solution

A parallelogram $ABCD$ in which bisectors of angles A, B, C, D intersect at P, Q, R, S to form a quadrilateral $PQRS$.

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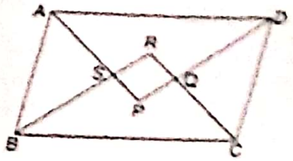


Since, $ABCD$ is a parallelogram, so $AD \parallel BC$ and transversal AB intersects them at A and B, respectively.

$$\angle A + \angle B = 180^\circ \Rightarrow \frac{1}{2}\angle A + \frac{1}{2}\angle B = 90^\circ$$

$$\Rightarrow \quad \angle BAS + \angle ABS = 90^\circ \qquad \ldots(i)$$

[AS and BS are bisectors of $\angle A$ and $\angle B$, respectively]

$$\angle BAS + \angle ABS + \angle ASB = 180^\circ \Rightarrow 90^\circ + \angle ASB = 180^\circ$$

$$\Rightarrow \quad \angle ASB = 90^\circ \Rightarrow \angle RSP = 90^\circ$$

[$\angle ASB$ and $\angle RSP$ are vertically opposite angles]

Similarly, $\angle SRQ = 90^\circ$, $\angle RQP = 90^\circ$ and $\angle SPQ = 90^\circ$.

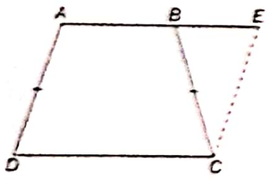
Hence, $PQRS$ is rectangle.

Example 28

$ABCD$ is a trapezium in which $AB \parallel CD$ and $AD = BC$. Show that:

(i) $\angle A = \angle B$
(ii) $\angle C = \angle D$
(iii) $\Delta ABC \cong \Delta BAD$
(iv) Diagonal AC = diagonal BD.

Solution



(i) Extend AB and draw a line through C parallel to DA intersecting AB produced at E.

Since, $AD \parallel CE$ and transversal AE cuts them at A and E, respectively.

$\therefore \quad \angle A + \angle E = 180^\circ \Rightarrow 180^\circ - \angle E = \angle A$

Since, $AB \parallel CD$ and $AD \parallel CE$

$\therefore$ $AECD$ is a parallelogram.

$\Rightarrow AD = CE \Rightarrow BC = CE$

Thus, in ΔBCE

$BC = CE \Rightarrow \angle CBE = \angle CEB \Rightarrow 180^\circ - \angle B = \angle E$

$\Rightarrow 180^\circ - \angle E = \angle B \Rightarrow \angle A = \angle B$.

(ii) Consecutive interior angles on the same side of a transversal are supplementary.

$\therefore \quad \angle A + \angle D = 180^\circ$ and $\angle E + \angle C = 180^\circ$

$\Rightarrow \quad \angle A + \angle D = \angle E + \angle C$

$\Rightarrow \quad \angle B + \angle D = \angle E + \angle C \Rightarrow \angle D = \angle C$ [$\angle B = \angle E$]

(iii) In ΔABC and ΔBAD

$AB = BA$

$\angle B = \angle A$ and $BC = AD$

So, by SAS congruence criterion

$\Delta ABC \cong \Delta BAD$

(iv) Since, $\Delta ABC \cong \Delta BAD$

$AC = BD$

Hence, diagonal AC = diagonal BD.

Example 29

If the medians of a ΔABC intersect at G, show that $\text{ar}(\Delta AGB) = \text{ar}(\Delta AGC) = \text{ar}(\Delta BGC) = \frac{1}{3}\text{ar}(\Delta ABC)$.

Solution

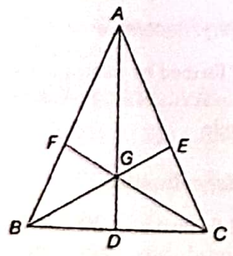
Given: A ΔABC and its medians AD, BE and CF intersect at G.

To prove: $\text{ar}(\Delta AGB) = \text{ar}(\Delta AGC) = \text{ar}(\Delta BGC) = \frac{1}{3}\text{ar}(\Delta ABC)$.

Proof: A median of a triangle divides it into two triangles of equal area.

In ΔABC, AD is the median.

$\therefore \quad \text{ar}(\Delta ABD) = \text{ar}(\Delta ACD) \qquad \ldots(i)$



In ΔGBC, GD is the median.

$\therefore \quad \text{ar}(\Delta GBD) = \text{ar}(\Delta GCD) \qquad \ldots(ii)$

Subtract equation (ii) from (i), we get

$$\text{ar}(\Delta ABD) - \text{ar}(\Delta GBD) = \text{ar}(\Delta ACD) - \text{ar}(\Delta GCD)$$

$\therefore \quad \text{ar}(\Delta AGB) = \text{ar}(\Delta AGC) \qquad \ldots(iii)$

Similarly, $\text{ar}(\Delta AGB) = \text{ar}(\Delta BGC) \qquad \ldots(iv)$

From (iii) and (iv)

$\text{ar}(\Delta AGB) = \text{ar}(\Delta AGC) = \text{ar}(\Delta BGC)$

But, $\text{ar}(ABC) = \text{ar}(\Delta AGB) + \text{ar}(\Delta AGC) + \text{ar}(\Delta BGC)$

$= 3\ \text{ar}\ (\Delta AGB)$

$\therefore \quad \text{ar}(\Delta AGB) = \frac{1}{3}\text{ar}(\Delta ABC)$.

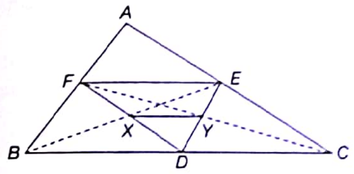
Hence, $\text{ar}(\Delta AGB) = \text{ar}(\Delta AGC) = \text{ar}(\Delta BGC) = \frac{1}{3}\text{ar}(\Delta ABC)$.

Example 30

In the adjoining figure, D, E, F are the mid-points of the sides BC, CA and AB of ΔABC. If BE and DF intersect at X while CF and DE intersect at Y, prove that $XY = \frac{1}{4}BC$.

Solution

In ΔABC, F and E are the mid-points of AB and AC, respectively.



$\therefore \quad FE \parallel BC$ and $FE = \frac{1}{2}BC = BD$.

$\therefore \quad FE \parallel BD$ and $FE = BD$.

So, $BDEF$ is a parallelogram whose diagonals BE and DF intersect each other at X.

$\therefore$ X is the mid-point of DF.

Similarly, Y is the mid-point of DE.

Thus, in ΔDEF, X and Y are the mid-points of DF and DE, respectively.

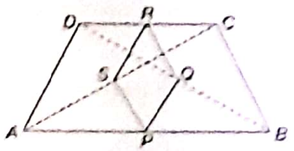
So, $XY \parallel FE$ and $XY = \frac{1}{2}FE = \frac{1}{2} \times \frac{1}{2}BC = \frac{1}{4}BC$.

Example 31

In the adjoining figure, $ABCD$ is a trapezium in which $AB \parallel DC$ and $AD = BC$. If P, Q, R, S be, respectively, the mid-points of BA, BD and CD, CA, then show that $PQRS$ is a rhombus.

Solution

In ΔBDC, Q and R are the mid-points of BD and CD, respectively.



$\therefore \quad QR \parallel BC$ and $QR = \frac{1}{2}BC$.

Similarly, $PS \parallel BC$ and $PS = \frac{1}{2}BC$.

$\therefore \quad PS \parallel QR$ and $PS = QR$ [Each equal to $\frac{1}{2}BC$]

$\therefore$ $PQRS$ is a parallelogram.

In ΔACD, S and R are the mid-points of AC and CD respectively.

$\therefore \quad SR \parallel AD$ and $SR = \frac{1}{2}AD = \frac{1}{2}BC$ [$\because AD = BC$]

$\therefore \quad PS = QR = SR = PQ$

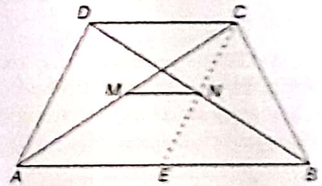
Hence, $PQRS$ is a rhombus.

Example 32

Prove that the line segment joining the mid-points of the diagonals of a trapezium is parallel to the parallel sides and equal to half of their difference.

Solution

Let $ABCD$ be a trapezium in which $AB \parallel DC$, and let M and N be the mid-points of the diagonal AC and BD, respectively.



Join CN and produce it to meet AB at E.

In ΔCDN and ΔEBN, we have

$DN = BN$ [$\because N$ is mid-point of BD]

$\angle DCN = \angle BEN$ [Alt. Int. $\angle$]

$\angle CDN = \angle EBN$ [Alt. Int. $\angle$]

$\therefore \quad \Delta CDN \cong \Delta EBN$ [AAS-criteria]

$\therefore \quad DC = EB$ and $CN = NE$ [CPCT]

Thus, in ΔCAE, the points M and N are the mid-points of AC and CE respectively.

$\therefore MN \parallel AE$ and $MN = \frac{1}{2} AE \Rightarrow MN \parallel AB \parallel DC$

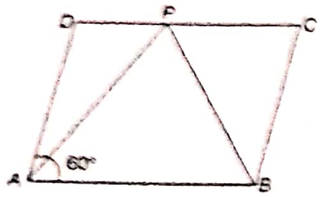
and

$$MN = \frac{1}{2} AE = \frac{1}{2} (AB - EB) = \frac{1}{2} (AB - DC)$$

$[\because EB = DC]$

Know Your Progress - 4

1. In the following figure, $ABCD$ is a parallelogram in which $\angle A = 60°$. If the bisectors of $\angle A$ and $\angle B$ meet at P, prove that $\angle APB = 90°$. Also, prove that $AD = DP$, $PC = BC$, and $DC = 2AD$.

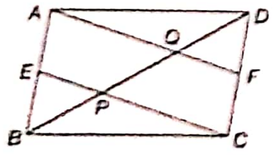


2. In ΔABC, AD is the median through A and E is the mid-point of AD. BE produced meets AC in F. Prove that $AF = \frac{1}{3} AC$.

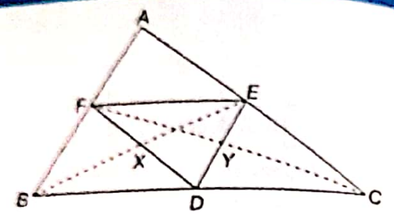
3. Let $ABCD$ be a trapezium in which $AB \parallel DC$, and let M and N be the mid-points of the diagonals AC and BD, respectively. If $AB = 15$ cm and $CD = 5$ cm, then find MN.

4. Prove that quadrilateral formed by line segments joining the mid-points of the sides of an isosceles trapezium is a rhombus.

5. In figure, $ABCD$ is a parallelogram. E and F are the mid-point of the sides AB and CD, respectively. Prove that the line segments AF and CE trisect (divide into three equal parts) the diagonal BD.

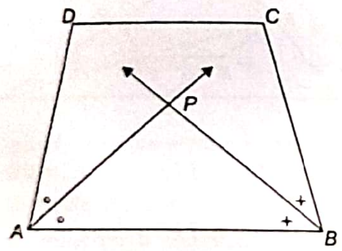


6. In the adjoining figure, D, E, F are the mid-points of the sides BC, CA and AB of ΔABC. If BE and DF intersect at X while CF and DE intersect at Y, prove that $XY = \frac{1}{4} BC$.



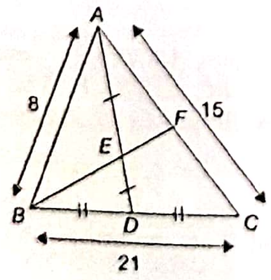
7. If the angle bisectors of $\angle DAB$ and $\angle CBA$ of any quadrilateral $ABCD$ intersects at the Point P, then prove that

$$2\angle APB = \angle BCD + \angle BDC$$



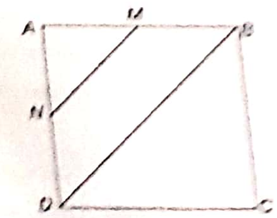
8. If the line segment joining the mid-point of the consecutive side of quadrilateral $ABCD$ form a rectangle then $ABCD$ must be
   (A) rhombus (B) square
   (C) kite (D) all of the above

9. In ΔABC, AD is median and E is the mid-point of AD. If BE is extended, it meets AC in F. $AB = 8$ cm, $BC = 21$ cm, and $AC = 15$ cm, then $AF = ?$



(A) 7 cm (B) 3 cm
(C) 12 cm (D) 5 cm

10. In the given figure, $ABCD$ is a square in which M and N are the mid-point of the sides $\overline{AB}$ and $\overline{AD}$ respectively. How many square units is the area of $MNDB$ quadrilateral, if each side of the square is 2 units long?

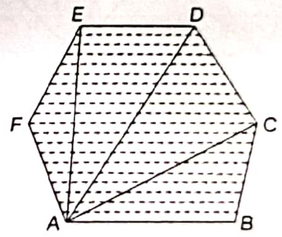


(A) 1.0 (B) 1.5
(C) 2.0 (D) 2.5

Area of Parallelograms and Triangles

Polygon Region

Polygon region can be expressed as the union of a finite number of triangular regions in a plane such that if two of these intersect, their intersection is either a point or a line segment. It is the shaded portion including its sides as shown in the figure.



Area Axioms

Every polygonal region R has an area, measured in square units and denoted by ar(R).

(i) **Congruent area axiom:** If R_1 and R_2 be two regions such that $R_1 \cong R_2$ then ar(R_1) = ar (R_2).

(ii) **Area addition axiom:** If R_1 and R_2 are two polygonal regions, whose intersection is a finite number of points and line segments such that $R = R_1 \cup R_2$, then ar(R) = ar(R_1) + ar(R_2).

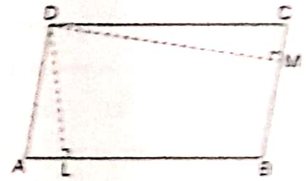
(iii) **Rectangular area axiom:** If AB = a metre and $AD = b$ metre then, ar (rectangular region $ABCD$) = ab sq.m.

Area of a Parallelogram

Base and altitude of a Parallelogram:

(i) **Base:** Any side of a parallelogram can be called its base.

(ii) **Altitude:** The length of the line segment that is perpendicular to the base from the opposite side is called the altitude or height of the parallelogram corresponding to the given base.



(a) DL is the altitude of $\parallel^{gm}$ $ABCD$, corresponding to the base AB.

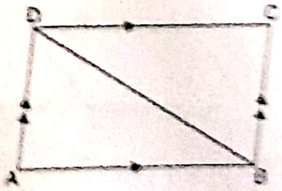
(b) DM is the altitude of $\parallel^{gm}$ $ABCD$, corresponding to the base BC.

Theorem: A diagonal of a parallelogram divides it into two triangles of equal area.

Given: A parallelogram $ABCD$ whose one of the diagonals is BD.

To prove: ar (ΔABD) = ar(ΔCDB).

Proof:



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In ΔABD and ΔCDB

$AB = DC$ [Opposite sides of a ||gm]

$AD = BC$ [Opposite sides of a ||gm]

$BD = BD$ [Common side]

$\therefore \Delta ABD \cong \Delta CDB$ [By SSS congruency]

$\therefore ar(\Delta ABD) = ar(\Delta CDB)$ Hence proved.

Theorem: Parallelograms on the same base and between the same parallels are equal in area.

Given: Two ||^{gm} $ABCD$ and $ABEF$ on the same base AB and between the same parallels AB and FC.

To prove: $ar(||^{gm} ABCD) = ar(||^{gm} ABEF)$

Proof: In ΔADF and ΔBCE, we have

$AD = BC$ [Opposite sides of a $||^{gm}$]

$AF = BE$ [Opposite sides of a $||^{gm}$]

$\angle DAF = \angle CBE$ [$\because AD \,||\, BC$ and $AF \,||\, BE$]

[Angle between AD and AF = Angle between BC and BE]

$\therefore \Delta ADF \cong \Delta BCE$ [By SAS congruency]

$\therefore ar(\Delta ADF) = ar(\Delta BCE)$ ...(i)

$\therefore ar(||^{gm} ABCD) = ar(ABED) + ar(\Delta BCE)$

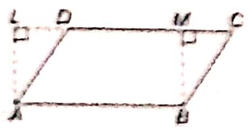
$= ar(ABED) + ar(\Delta ADF)$ [Using (i)]

$= ar(||^{gm} ABEF)$.

Hence, $ar(||^{gm} ABCD) = ar(||^{gm} ABEF)$.

Hence proved.

Theorem: The area of parallelogram is the product of its base and the corresponding altitude.



Given: A $||^{gm} ABCD$ in which AB is the base and AL is the corresponding height.

To prove: Area ($||^{gm} ABCD$) = $AB \times AL$

Construction: Draw $BM \perp DC$, so that rectangle $ABML$ is formed.

Proof: $||^{gm} ABCD$ and rectangle $ABML$ are on the same base AB and between the same parallel lines AB and LC.

$\therefore ar(||^{gm} ABCD) = ar(\text{rectangle } ABML) = AB \times AL$

$\therefore$ area of a $||^{gm}$ = base × height. Hence proved.

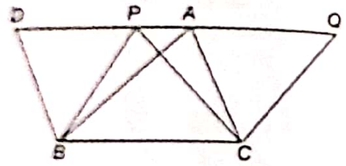
Area of a Triangle

Theorem: Two triangles on the same base (or equal bases) and between the same parallels are equal in area.

Given: Two triangles ABC and PBC on the same base BC and between the same parallel lines BC and AP.

To prove: $ar(\Delta ABC) = ar(\Delta PBC)$.

Construction: Through B, draw $BD \,||\, CA$ intersecting AP produced in D and through C, draw $CQ \,||\, BP$, intersecting PA produced in Q.



Proof: $BD \,||\, CA$ [By construction]

And, $BC \,||\, DA$ [Given]

$\therefore$ Quadrilateral $BCAD$ is a parallelogram.

Similarly, Quadrilateral $BCQP$ is a parallelogram.

Now, parallelogram $BCQP$ and $BCAD$ are on the same base BC, and between the same parallels.

$\therefore ar(||^{gm} BCQP) = ar(||^{gm} BCAD)$ ...(i)

Diagonals of a parallelogram divides it into two triangles of equal area.

$\therefore ar(\Delta PBC) = \frac{1}{2} ar(||^{gm} BCQP)$ ...(ii)

And $ar(\Delta ABC) = \frac{1}{2} ar(||^{gm} BCAD)$ ...(iii)

Now, $ar(||^{gm} BCQP) = ar(||^{gm} BCAD)$ [From (i)]

$\Rightarrow \frac{1}{2} ar(||^{gm} BCAD) = \frac{1}{2} ar(||^{gm} BCQP)$

Hence, $ar(\Delta ABC) = ar(\Delta PBC)$ [Using (ii) and (iii)]

Hence proved.

Theorem: If a triangle and a parallelogram are on the same base and between the same parallels, the area of the triangle is equal to half of the parallelogram.

Given: A ΔABC and a parallelogram $BCDE$ on the same base BC and between the same parallels BC and AD.

To prove: $ar(\Delta ABC) = \frac{1}{2}$ ar (parallelogram $BCDE$).

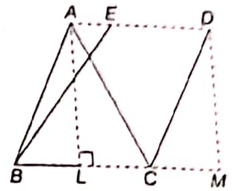
Construction: Draw $AL \perp BC$ and $DM \perp BC$, meeting BC producted in M.

Proof: Since E and D are colinear and $BC \,||\, AD$

$\therefore AL = DM$ ...(i)

[$\because$ distance between parallel lines is always same]

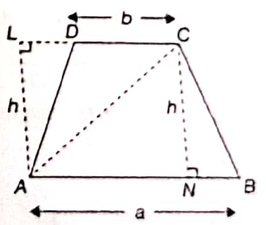
Now, $ar(\Delta ABC) = \frac{1}{2}(BC \times AL)$



$\Rightarrow ar(\Delta ABC) = \frac{1}{2}(BC \times DM)$ [$\because AL = DM$ (from (i))]

$\Rightarrow ar(\Delta ABC) = \frac{1}{2}$ ar(parallelogam $BCDE$).

Theorem: The area of a trapezium is half the product of its height and the sum of the parallel sides.



Given: Trapezium $ABCD$ in which $AB \,||\, DC$, $AL \perp DC$, $CN \perp AB$ and $AL = CN = h$ (say), $AB = a$, $DC = b$.

To prove: ar(trapezium $ABCD$) = $\frac{1}{2} h \times (a + b)$.

Construction: Join AC.

Proof: $\because AC$ is a diagonal of quad. $ABCD$.

$\therefore ar(\text{trapezium } ABCD) = ar(\Delta ABC) + ar(\Delta ACD)$

$= \frac{1}{2} b \times a + \frac{1}{2} b \times h = \frac{1}{2} h(a + b)$.

Hence proved.

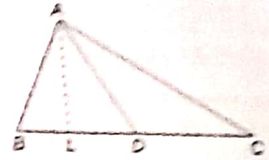
Theorem: Median of a triangle divides it into two triangles of equal area.

Given: A ΔABC in which AD is the median.

To prove: $ar(\Delta ABD) = ar(\Delta ADC)$.

Construction: Draw $AL \perp BC$.

Proof: Since AD is the median of ΔABC. Therefore, D is the mid-point of BC.



$\Rightarrow ar(\Delta ABD) = \frac{1}{2}(BD \times AL)$ ...(i)

$\Rightarrow ar(\Delta ADC) = \frac{1}{2}(CD \times AL)$ ...(ii)

[$\because BD = CD$, AD is the median of ΔABC]

From (i) and (ii)

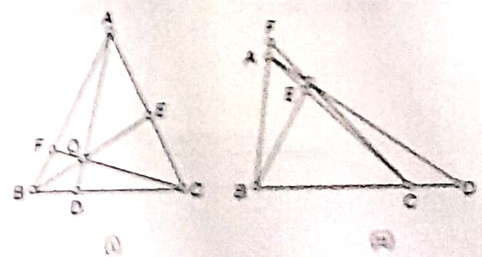
$ar(\Delta ABD) = ar(\Delta ADC)$.

Hence proved.

Some Important Theorem

Theorem (Ceva) If points D, E, F are taken on the sides BC, CA, AB of ΔABC so that the lines AD, BE, CF are concurrent at a point O, then

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1 \quad \text{...(1)}$$



(i) (ii)

Proof: There are two possible cases depending on whether O lies inside or outside ΔABC. First note that in figure (i) all the ratios $\frac{BD}{DC}$ etc. are positive, while in figure (ii) exactly two are negative and so the product in (1) is positive in all cases. Hence we now ignore the signs of the segments and prove that numerical value of the product in (1) is 1.

In figure (i) and (ii), since the triangle ABD, ADC have the same altitude and also the triangle OBD, ODC have the same altitude, we get

$$\frac{BD}{DC}=\frac{\Delta ABD}{\Delta ADC}=\frac{\Delta OBD}{\Delta ODC}=\frac{\Delta ABD-\Delta OBD}{\Delta ADC-\Delta ODC}=\frac{\Delta ABO}{\Delta CAO}$$

Similarly,

$$\frac{CE}{EA}=\frac{\Delta BCO}{\Delta ABO},\ \frac{AF}{FB}=\frac{\Delta CAO}{\Delta BCO}$$

Multiplying these three equations, we get (1).

Theorem (Converse of Ceva's theorem). If three points D, E, F taken on the sides BC, CA, AB of ΔABC are such that equation (1) holds, then AD, BE, CF are concurrent.

Proof. Suppose AD and BE meet in O and meets AB in 'F'. Then since AD, BE, CF' are concurrent, we get by Ceva's Theorem,

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF'}{F'B}=1.$$

Hence by (1) we get $\frac{AF}{FB}=\frac{AF'}{F'B}$

Hence $\frac{AF}{AF+FB}=\frac{AF'}{AF'+F'B}$ or $\frac{AF}{AB}=\frac{AF'}{AB}$

Thus $AF=AF'$ and so the points F and F' coincide. Hence CF also passes through O

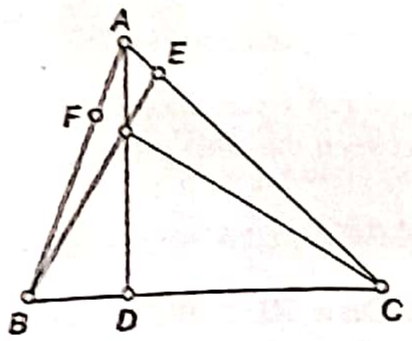
Trigonometric form of Ceva's Theorem: Let points D, E, F be taken on the sides BC, CA, AB of ΔABC. Then the lines AD, BE, CF are concurrent if and only if

$$\frac{\sin\angle BAD}{\sin\angle DAC}\cdot\frac{\sin\angle CBE}{\sin\angle EBA}\cdot\frac{\sin\angle ACF}{\sin\angle FCB}=1.$$

Corollary 1. The medians of a triangle are concurrent.

[For in this case, figure (i), $BD/DC=1$ etc. and so (1) holds.]

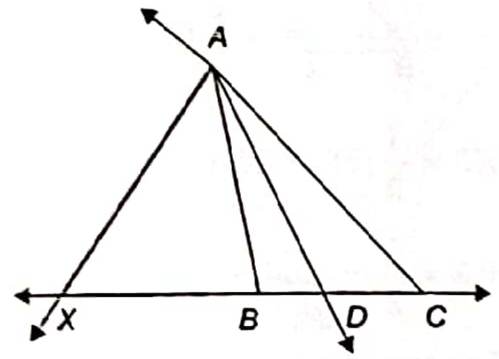
Corollary 2. The altitudes of a triangle are concurrent.



(iii)

[For acute-angled triangle ABC, figure (iii) $BE\perp CA$ and $CF\perp AB$ and so $\Delta AEB\sim\Delta AFC$ as $\angle A$ is common to them. Hence $\frac{AE}{AF}=\frac{AB}{AC}$ Similarly, if $AD\perp BC$, then $\frac{BF}{BD}=\frac{BC}{BA}$ and $\frac{CD}{CE}=\frac{CA}{CB}$ so $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=\frac{AB}{BC}\cdot\frac{BC}{CA}\cdot\frac{CA}{AB}=1$. And so (1) holds. A similar proof holds for obtuse angled triangles]

Corollary 3. The internal bisectors of the angle of a triangle are concurrent.



(iv)

In ΔABC, the internal bisector AD of $\angle A$ divides the opposite side BC internally in the ratio $\frac{BD}{DC}=\frac{AB}{AC}$; the external bisector AX of $\angle A$ divides BC externally in the ratio $\frac{BX}{XC}=-\frac{AB}{AC}$. (see figure iv)

[For in this case, figure $\frac{BD}{DC}=\frac{AB}{AC}=\frac{c}{b}$, etc., where a, b, c, respectively, denotes the lengths of the sides BC, CA, AB of ΔABC. Hence

$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=\frac{c}{b}\cdot\frac{a}{c}\cdot\frac{b}{a}=1$ and so (1) holds]

Corollary 4. The external bisectors of any two angles of a triangle and the internal bisector of the third angles are concurrent.

[For in this case, (Figure iv)

$\frac{BD}{DC}=\frac{c}{b}$, $\frac{CE}{EA}=\frac{-a}{c}$, $\frac{AF}{FB}=\frac{-b}{a}$ and so (1) holds.]

Corollary 5. If the incircle of ΔABC touches BC, CA, AB at X, Y, Z respectively, then AX, BY, CZ are concurrent (the point of concurrence is called the Gergonne point of ΔABC)

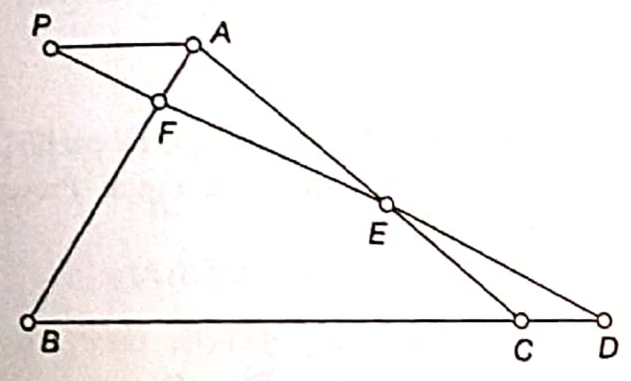
[For in this case, (figure) $BX=BZ$, etc. and so

$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=\frac{BX}{CY}\cdot\frac{CY}{AZ}\cdot\frac{AZ}{BX}=1$, and so (1) holds]

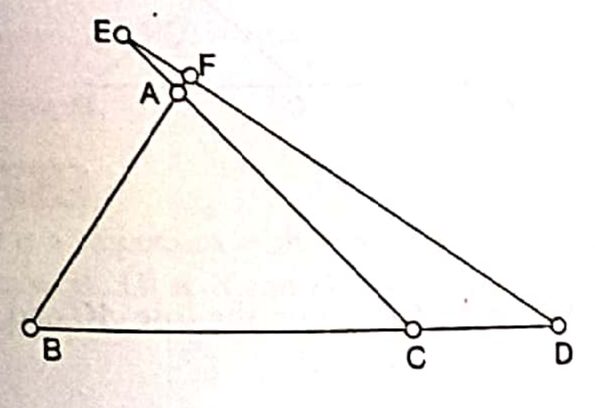
Theorem (Menelaus) If a transversal cut the sides BC, CA, AB (Suitably extended) of ΔABC in point D, E, F, respectively, then

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=-1 \qquad \ldots(2)$$

Proof. There are two cases depending on whether the transversal cuts one side externally or all three sides externally. Note that only one ratio (as in Figure (v)) or else all the three ratios (as in figure (vi)) in (2) are negative. Thus, the product in (2) is always negative. Hence, we now ignore the signs of the segments and prove that the numerical value of the product in (2) is 1.



(v)



(vi)

With the notation of figure (v) draw AP parallel to DB meeting DF in P. Then, since $\Delta APF\sim\Delta BDF$,

$\frac{AF}{FB}=\frac{PA}{BD}$ and since $\Delta CDE-\Delta APE$, $\frac{DC}{AP}=\frac{CE}{EA}$.

Hence

$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=\frac{BD}{DC}\cdot\frac{DC}{AP}\cdot\frac{PA}{BD}=1$, numerically.

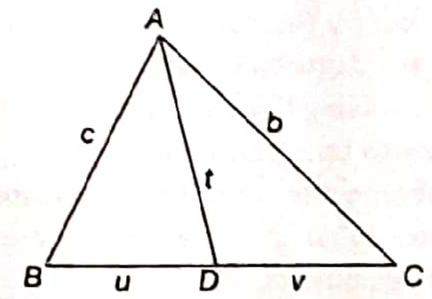
Theorem (Converse of Menelaus' Theorem): If points D, E, F are taken on the sides of ΔABC such that equation (2) holds, then D, E, F are collinear.

Proof. Let DE meet AB in F'. Then by Menelaus' theorem, we get

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF'}{F'B}=-1.$$

Hence, by (2) we get $\frac{AF}{FB}=\frac{AF'}{F'B}$ and so $F=F'$ Hence F lies on DE, as required.

Theorem (Stewart's)



(i) If u, v are the actual length of BD, DC respectively then

$$t^2=\frac{ub^2+vc^2}{u+v}-uv$$

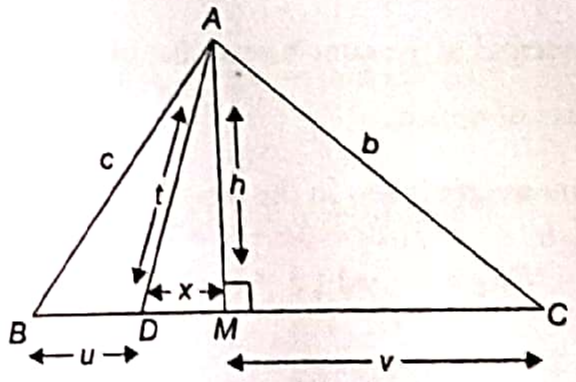
(ii) If $BD:DC=u:v$ then, $ub^2+vc^2=(u+v)t^2+u\cdot DC^2+v\cdot BD^2$

Note: Apollonius thoerem is special case of Stewart theorem.

Stewart's Theorem (Proof):

Let $\angle B$ and $\angle C$ are acute angles and $u<v$

Draw $AM\perp BC$



In ΔADM, ΔABM and ΔACM

$$c^2 = h^2 + x^2 \quad (1)$$
$$b^2 = h^2 + (r - x)^2 \quad (2)$$
$$c^2 = h^2 + (r + x)^2 \quad (3)$$
$$\Rightarrow \quad b^2u = h^2u^2 + r^2u + x^2 \quad (4)$$
$$\Rightarrow \quad c^2v = h^2v + u^2v + x^2v + 2uxv \quad \text{[from equation (3)]} \quad (5)$$

On adding equations (4) and (5)

$$b^2u + c^2v = h^2v + h^2u + r^2u + u^2v + x^2(u + v)$$
$$\Rightarrow \quad b^2u + c^2v = (u + v)(h^2 + ur + x^2)$$

From equation (1)

$$b^2u + c^2v = (u + v)(ur + r^2)$$
$$r^2 = \frac{b^2u \quad c^2v}{u \quad v} \quad \text{or}$$

Mass Point Geometry

Mass Point Geometry (MPG) is a technique borrowed from the concept of physics which was apply to solve many complex problems involving triangles and intersecting lines. The technique is easy to understand and apply.

MPG Technique also simplifies calculations of ratios in geometric figures in an intuitive way by merging algebra, geometry and basic physics.

The method is as simple as balancing a see-saw. The basic idea is that of a see-saw with weights at each end. The see-saw will balance if the product of the mass and its distance to the fulcrum is the same for each mass.

For example, if a square pyramid is 3m from the fulcrum, then a cube of the same base must be located 1m on the other side of fulcrum for see-saw to balance.

Square pyramid of the same base as that of a cube has $\frac{1^{st}}{3}$ of the volume of square.

Hence, the weight is also in the ratio 1 : 3, say $x : 3x$
Total weight at the pivot $= 3x + x = 4x$
Distance $\times$ Weight $= (3m)(x) = (1m)(3x)$

i.e., Distance weight product $=$ (3mx) on both sides of the fulcrum therefore the see-saw is balanced.

$$\therefore \quad m_1 l_1 = m_2 l_2$$

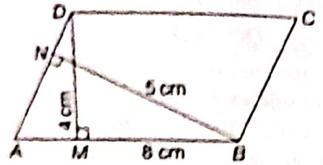
i.e., $$\frac{m_1}{m_2} = \frac{l_2}{l_1}$$

Example 33

In a parallelogram $ABCD$, $AB = 8$ cm. The altitudes corresponding to sides AB and AD are, respectively, 4 cm and 5 cm. Find AD.

Solution

Area of a $\|^{gm}$ = Base $\times$ corresponding altitude

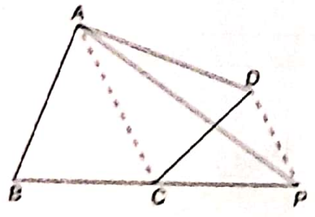


$\therefore$ Area of parallelogram $ABCD = AD \times BN = AB \times DM$

$$\Rightarrow \quad AD \times 5 = 8 \times 4 \Rightarrow AD = \frac{8 \times 4}{5} = 6.4 \text{ cm.}$$

Example 34

$ABCD$ is a quadrilateral. A line through D, parallel to AC, meets BC produced in P as shown in figure. Prove that ar (ΔABP) = ar(quad. $ABCD$).



Solution

Since Δs ACP and ACD are on the base AC and between the same parallel's AC and DP

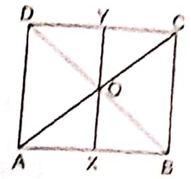
$$\therefore \quad \text{ar}(\Delta ACP) = \text{ar}(\Delta ACD)$$
$$\Rightarrow \quad \text{ar}(\Delta ACP) + \text{ar}(\Delta ABC) = \text{ar}(\Delta ACD) + \text{ar}(\Delta ABC)$$
$$\Rightarrow \quad \text{ar}(\Delta ABP) = \text{ar}(\text{quad. } ABCD)$$

Example 35

The diagonals of a parallelogram $ABCD$ intersect in O. A line through O meets AB in X and the opposite side CD in Y. Show that ar (quadrilateral $AXYD$) $= \frac{1}{2}$ ar(parallelogram $ABCD$).

Solution

$\because$ AC is a diagonal of the parallelogram $ABCD$



$$\text{ar}(\Delta ACD) = \frac{1}{2}\text{ar}(\|^{gm} ABCD) \quad \ldots(i)$$

Now, in Δs AOX and COY,

$$AO = CO$$

[$\because$ Diagonals of a parallelogram bisect each other]

$$\angle AOX = \angle COY \quad \text{[Vertically opposite } \angle\text{s]}$$
$$\angle OAX = \angle OCY \quad \text{[Alternate interior } \angle\text{s]}$$

[$\because$ $AB \| DC$ and transversal AC intersects them]

$$\therefore \quad \Delta AOX \cong \Delta COY \quad \text{[By } ASA \text{ congruency]}$$
$$\therefore \quad \text{ar}(\Delta AOX) = \text{ar}(\Delta COY) \quad \ldots(ii)$$

Adding ar(quad. $AOYD$) to both sides of (ii), we get

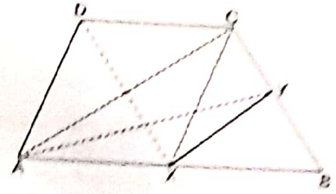
$$\text{ar(quad. } AOYD) + \text{ar}(\Delta AOX) = \text{ar(quad. } AOYD) + \text{ar}(\Delta COY)$$
$$\Rightarrow \quad \text{ar(quad. } AXYD) = \text{ar}(\Delta ACD) = \frac{1}{2}\text{ar}(\|^{gm} ABCD)$$

Hence proved.

Example 36

$ABCD$ is a trapezium with $AB \| DC$. A line parallel to AC intersects AB at X and BC at Y. Prove that ar(ΔADX) = ar(ΔACY).

Solution



Join CX, DX, and AY.

Clearly, triangles ADX and ACX are on the same base AX and between the parallels AB and DC.

$$\therefore \quad \text{ar}(\Delta ADX) = \text{ar}(\Delta ACX) \quad \ldots(i)$$

Also, ΔACX and ΔACY are on the same base AC and between the parallels AC and XY.

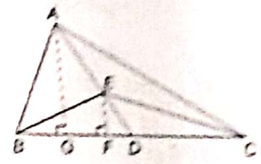
$$\therefore \quad \text{ar}(\Delta ACX) = \text{ar}(\Delta ACY) \quad \ldots(ii)$$

From (i) and (ii), we get

$$\text{ar}(\Delta ADX) = \text{ar}(\Delta ACY)$$

Example 37

In figure, E is any point on median AD of a ΔABC. Show that ar(ΔABE) = ar(ΔACE).



Solution

Construction: From A, draw $AG \perp BC$ and from E, draw $EF \perp BC$.

Proof: ar$(\Delta ABD) = \frac{BD \times AG}{2}$ and ar$(\Delta ADC) = \frac{DC \times AG}{2}$

But, $$BD = DC$$

[$\because$ D is the mid-point of BC, AD being the median]

$$\text{ar}(\Delta ABD) = \text{ar}(\Delta ADC) \quad \ldots(i)$$

Again, ar$(\Delta EBD) = \frac{BD \times EF}{2}$ and ar$(\Delta EDC) = \frac{DC \times EF}{2}$

But, $$BD = DC$$

$$\therefore \quad \text{ar}(\Delta EBD) = \text{ar}(\Delta EDC) \quad \ldots(ii)$$

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Subtracting (ii) from (i), we get

$\text{ar}(\Delta ABD) - \text{ar}(\Delta EBD) = \text{ar}(\Delta ADC) - \text{ar}(\Delta EDC)$

$\Rightarrow \quad \text{ar}(\Delta ABE) = \text{ar}(\Delta ACE)$.

Hence proved.

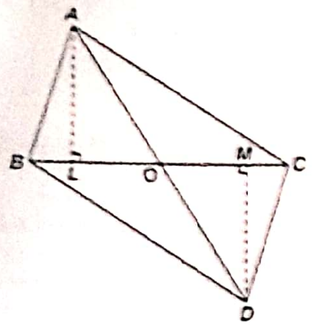
Example 38

Triangles ABC and DBC are on the same base BC, with A, D on opposite sides of the line BC, such that $\text{ar}(\Delta ABC) = \text{ar}(\Delta DBC)$. Show that BC bisects AD.

Solution

Construction: Draw $AL \perp BC$ and $DM \perp BC$.

Proof: $\text{ar}(\Delta ABC) = \text{ar}(\Delta DBC)$ [Given]



$\Rightarrow \quad \frac{BC \times AL}{2} = \frac{BC \times DM}{2}$

$\Rightarrow \quad AL = DM$ ...(i)

Now in Δs OAL and OMD

$AL = DM$ [From (i)]

$\Rightarrow \quad \angle ALO = \angle DMO$ [Each = 90°]

$\Rightarrow \quad \angle AOL = \angle MOD$ [Vertically opposite $\angle s$]

$\therefore \quad \Delta OLA \cong \Delta OMD$ [By AAS congruency]

$\therefore \quad OA = OD$ [By CPCT]

i.e., BC bisects AD.

Hence proved.

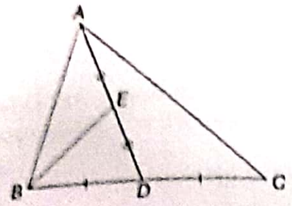
Example 39

ABC is a triangle in which D is the mid-point of BC and E is the mid-point of AD. Prove that the area of $\Delta BED = \frac{1}{4}$ area of ΔABC.

Solution

Given: A ΔABC in which D is the mid-point of BC and E is the mid-point of AD.

To prove: $\text{ar}(\Delta BED) = \frac{1}{4}\text{ar}(\Delta ABC)$.



Proof: $\because$ AD is a median of ΔABC.

$\therefore \quad \text{ar}(\Delta ABD) = \text{ar}(\Delta ADC) = \frac{1}{2}\text{ar}(\Delta ABC)$ ...(i)

[$\because$ Median of a triangle divides it into two triangles of equal area]

Again,

$\because$ BE is a median of ΔABD.

$\therefore \quad \text{ar}(\Delta BEA) = \text{ar}(\Delta BED) = \frac{1}{2}\text{ar}(\Delta ABD)$

[$\because$ Median of a triangle divides it into two triangles of equal area]

And $\quad \text{ar}(\Delta BED) = \frac{1}{2}\text{ar}(\Delta ABD) = \frac{1}{2} \times \frac{1}{2}\text{ar}(\Delta ABC)$

[From (i)]

$\therefore \quad \text{ar}(\Delta BED) = \frac{1}{4}\text{ar}(\Delta ABC)$.

Hence proved.

Example 40

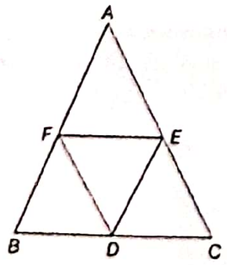
D, E and F are respectively the mid points of the sides BC, CA and AB of a ΔABC. Show that:

(i) $BDEF$ is a parallelogram

(ii) $\text{ar}(||^{gm} BDEF) = \frac{1}{2}\text{ar}(\Delta ABC)$

(iii) $\text{ar}(\Delta DEF) = \frac{1}{4}\text{ar}(\Delta ABC)$

Solution



(i) In ΔABC,

$\because$ F is the mid-point of side AB and E is the mid-point of side AC.

$\therefore$ $BF \parallel BD$ [$\because$ Line joining the mid-points of any two sides of a Δ is parallel to the third side.]

Similarly, $BD \parallel FE$.

Hence, $BDEF$ is a parallelogram.

Hence proved.

(ii) Similarly, we can prove that $AFDE$ and $FDCE$ are parallelograms.

$\because$ FD is a diagonal of parallelogram $BDEF$.

$\therefore \quad \text{ar}(\Delta FBD) = \text{ar}(\Delta DEF)$ ...(i)

Similarly, $\quad \text{ar}(\Delta FAE) = \text{ar}(\Delta DEF)$ ...(ii)

And, $\quad \text{ar}(\Delta DCE) = \text{ar}(\Delta DEF)$ ...(iii)

From above equations, we have

$\text{ar}(\Delta FBD) = \text{ar}(\Delta FAE) = \text{ar}(\Delta DCE) = \text{ar}(\Delta DEF)$

and $\text{ar}(\Delta FBD) + \text{ar}(\Delta DCE) + \text{ar}(\Delta DEF) + \text{ar}(\Delta FAE) = \text{ar}(\Delta ABC)$

$\Rightarrow \quad 2\,[\text{ar}(\Delta FBD) + \text{ar}(\Delta DEF)] = \text{ar}(\Delta ABC)$

[By using (i) and (iii)]

$\Rightarrow \quad 2\,[\text{ar}.(||^{gm} BDEF)] = \text{ar}(\Delta ABC)$

$\Rightarrow \quad \text{ar}(||^{gm} BDEF) = \frac{1}{2}\text{ar}(\Delta ABC)$.

(iii) Since, ΔABC is divided into four non-overlapping triangles FBD, FAE, DCE, and DEF.

$\therefore \quad \text{ar}(\Delta ABC) = \text{ar}(\Delta FBD) + \text{ar}(\Delta FAE) + \text{ar}(\Delta DCE) + \text{ar}(\Delta DEF)$

$\Rightarrow \quad \text{ar}(\Delta ABC) = 4\,\text{ar}(\Delta ABC)$

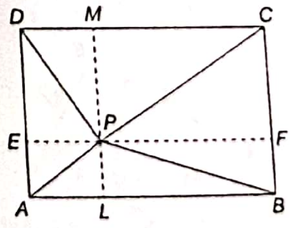
[Using (i), (ii) and (iii)]

$\Rightarrow \quad \text{ar}(\Delta DEF) = \frac{1}{4}\text{ar}(\Delta ABC)$.

Hence proved.

Example 41

In figure, P is a point in the interior of a rectangle $ABCD$. Show that



(i) $\text{ar}(\Delta APD) + \text{ar}(\Delta PBC) = \frac{1}{2}\text{ar}(\text{rectangle } ABCD)$

(ii) $\text{ar}(\Delta APD) + \text{ar}(\Delta PBC) = \text{ar}(\Delta APB) + \text{ar}(\Delta PCD)$

Solution

Construction: Draw $EPF \parallel AB \parallel CD$ and $LPM \parallel AD \parallel BC$.

Proof:

(i) $EPF \parallel AB$ and DA cuts them.

$\therefore \quad \angle DEP = \angle EAB = 90°$ [Corresponding angles]

$\therefore \quad PE \perp AD$

Similarly, $PF \perp BC$, $PL \perp AB$ and $PM \perp DC$.

$\therefore \quad \text{ar}(\Delta APD) + \text{ar}(\Delta BPC)$

$= \left(\frac{1}{2} \times AD \times PE\right) + \left(\frac{1}{2} \times BC \times PF\right)$

$= \frac{1}{2} AD \times (PE + PF)$ [$\because BC = AD$]

$= \frac{1}{2} \times AD \times EF = \frac{1}{2} \times AD \times AB$ [$\because EF = AB$]

$= \frac{1}{2} \times \text{ar}(\text{rectangle } ABCD)$.

(ii) $\text{ar}(\Delta APB) + \text{ar}(PCD)$

$= \left(\frac{1}{2} \times AB \times PL\right) + \left(\frac{1}{2} \times DC \times PM\right)$

$= \frac{1}{2} \times AB \times (PL + PM)$ [$\because DC = AB$]

$= \frac{1}{2} \times AB \times LM = \frac{1}{2} \times AB \times AD$ [$\because LM = AD$]

$= \frac{1}{2} \times \text{ar}(\text{rect. } ABCD)$.

$\therefore \quad \text{ar}(\Delta APD) + \text{ar}(PBC) = \text{ar}(\Delta APB) + \text{ar}(PCD)$

Hence proved.

Example 42

$ABCD$ is a parallelogram. X and Y are mid-points of BC and CD respectively.

Prove that area $(\Delta AXY) = \frac{3}{8}$ area(parallelogram $ABCD$).

Solution

Since X and Y are the mid-points of sides BC and CD, respectively, in ΔBCD.

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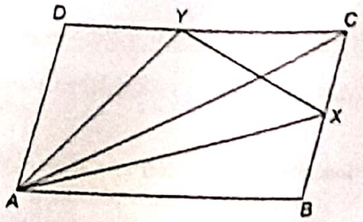


$\therefore$ $XY \parallel BD$ and $XY = \frac{1}{2} BD$.

$$\text{area } (\Delta CYX) = \frac{1}{4} \text{area}(\Delta DBC)$$

$$= \frac{1}{8} \text{area(parallelogram } ABCD) \quad \ldots(i)$$

[Since ar$(\Delta DBC) = \frac{1}{2}$ area (parallelogram $ABCD$)]

Parallelogram $ABCD$ and ΔABX are between the same parallels AD and BC and $BX = \frac{1}{2} BC$.

$\therefore$ area $(\Delta ABX) = \frac{1}{4}$ area (parallelogram $ABCD$) $\ldots(ii)$

Similarly, area $(\Delta AYD) = \frac{1}{4}$ area (parallelogram $ABCD$)

Now, ar(ΔAXY) = ar(parallelogram $ABCD$) $-$ [ar(ΔABX) + ar(ΔAYD) + ar(ΔCYX)]

$$\text{ar}(\Delta AXY) = \text{ar (parallelogram } ABCD) - \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{8}\right) \text{ar}(ABCD)$$

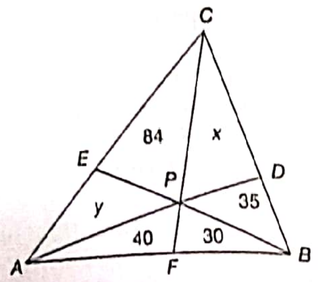
$$\text{ar}(\Delta AXY) = \left(1 - \frac{5}{8}\right) \text{ar(parallelogram } ABCD)$$

$$= \frac{3}{8} \text{ ar(parallelogram } ABCD).$$

Example 43

As shown in the figure, triangle ABC is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle ABC.

Solution



$\frac{[CAP]}{[FAP]} = \frac{CP}{FP} = \frac{[CBP]}{[FBP]}$ yields

$$\frac{84+y}{40} = \frac{x+35}{30}, \quad \ldots(1)$$

and $\frac{[CAP]}{[CDP]} = \frac{AP}{DP} = \frac{[BAP]}{[BDP]}$ yields

$$\frac{84+y}{x} = \frac{70}{35} = 2 \quad \ldots(2)$$

But $\frac{(2)}{(1)}$, it follows that $\frac{x}{40} = \frac{x+35}{60}$,

$\therefore$ $3x = 2x + 70$, i.e., $x = 70$.

Then by (2), $y = 140 - 84 = 56$.

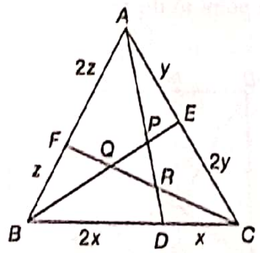
Thus, $[ABC] = 84 + 56 + 40 + 30 + 35 + 70 = 315$.

Example 44

On the sides BC, CA, and AB of ΔABC point D, E, F area takes in such a way that $\frac{DB}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = 2$. If the area of Δ determined by the line AD, BE, CF in $\frac{1}{K}$ times of area ΔABC find K.

Solution

In ΔADB, CF in cevian.



$$\frac{AR}{RD} \times \frac{DC}{CB} \times \frac{BF}{AF} = 1$$

$\therefore$

$$\frac{AR}{RD} = \frac{6}{1}$$

area $\Delta ABC = a$

$$\text{area } \Delta ADC = \frac{a}{3}$$

$$\text{area } \Delta ARC = \frac{6}{7}\left(\frac{a}{3}\right) = \frac{2}{7}a$$

Similarly, area ΔBQC = area $\Delta ABP = \frac{2a}{7}$

Adding area ΔARC + area ΔBQC + area ΔABP

$$= \frac{2a}{7} + \frac{2a}{7} + \frac{2a}{7} = \frac{6a}{7}$$

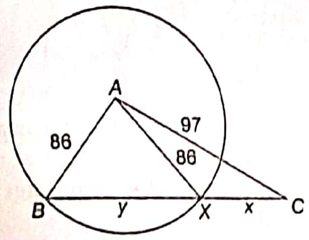
$\therefore$ area $\Delta PQR = a - \frac{6a}{7} = \frac{a}{7}$.

$\therefore$ $K = 7$.

Example 45

In ΔABC, $AB = 86$, $AC = 97$ a circle with centre A and Radius AB interest BC at B and X. BX and CX have integer length. What is the length of BC?

Solution



By Stewart Theorem

$$86^2 = \frac{x \cdot 86^2 + y \cdot 97^2}{x+y} - xy$$

$$x(x+y) = 33 \times 61 = 3 \times 11 \times 61$$

Note $x < x + y$

$$x + y < 86 + 97 = 183$$

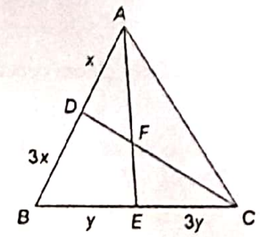
So possible solution $x = 33$, $x + y = 61$

Example 46

ABC is a triangle, D and E are interior points of sides, AB and BC, respectively, such that $\frac{AD}{DB} = \frac{1}{3}$, $\frac{DE}{EB} = \frac{1}{3}$, if AE and CD intersect at F. Find $\frac{CF}{FD}$.

Solution

Method – 1 By Menelaus theorem:



In ΔBCD by Menelaus theorem,

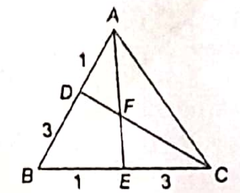
$$\frac{BE}{EC} \times \frac{CF}{FD} \times \frac{AD}{AB} = 1$$

$$\frac{y}{3y} \times \frac{CF}{FD} \times \frac{x}{4x} = 1$$

$$\frac{CF}{FD} = \frac{12}{1}$$

Method – 2 By Mass Point Geometry:

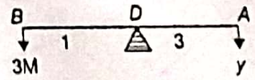
Let the mass of vertex (B) = $3M$



$$3M \times 1 = 3 \times x$$

$\Rightarrow$ $x = M$

$\therefore$ Mass at vertex (C) = M



$$3M \times 3 = 1 \times y$$

$\Rightarrow$ $y = 9M$

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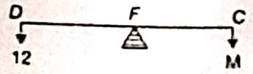
$\therefore$ Mass at vertex $(A) = 9M$

Total mass at $D = 3M + 9M = 12M$



$$12M \times DF = FC \times M$$

$$\Rightarrow \frac{CF}{DF} = \frac{12}{1}$$

Example 47

Prove that the medians of triangle are concurrent.

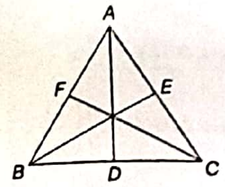
Solution

If D, E, F are the mid points of BC, CA, AB respectively than $DB = DC$; $CE = EA$; $AF = AB$

$\therefore$ $\frac{BD}{DC} = 1$; $\frac{CE}{EA} = 1$; $\frac{AF}{FB} = 1$

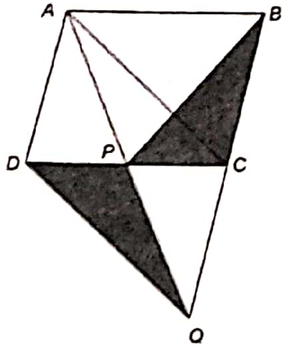
$\therefore$ $\frac{BD}{DC} \cdot \frac{CF}{EA} \cdot \frac{AF}{FB} = 1$

Thus medians AD, BF, CF are concurrent.

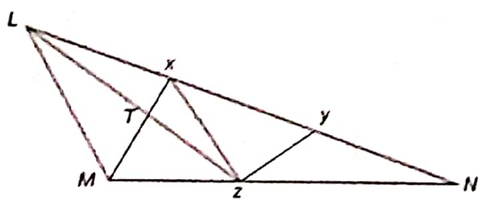


Know Your Progress - 5

1. $ABCD$ is a parallelogram and BC is produced to a point Q such that $AD = CQ$ (Figure). If AQ intersects DC at P, show that $ar(BPC) = ar(DPQ)$.



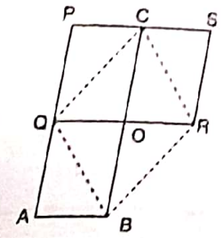
2. X and Y are points on the side LN of the triangle LMN such that $LX = XY = YN$. Through X, a line is drawn parallel to LM to meet MN at Z (See Figure). Prove that $ar(LZY) = ar(MZYX)$



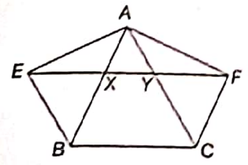
3. $ABCD$ is a parallelogram in which BC is produced to E such that $CE = BC$ (Figure). AE intersects CD at F. If $ar(DFB) = 3\ cm^2$, find the area of the parallelogram $ABCD$.

4. If the mid-points of the sides of a quadrilateral are joined in order, prove that the area of the parallelogram so formed will be half of the area of the given quadrilateral (Figure).

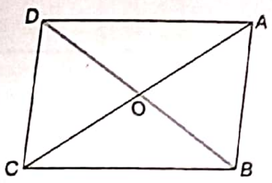
5. In the adjoining figure, $PQRS$ and $PABC$ are two parallelograms of equal area. Prove that $QC \parallel BR$.



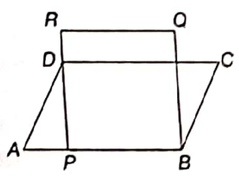
6. In the given figure, XY is a line parallel to side BC of a ΔABC. $BE \parallel AC$ and $CF \parallel AB$ meet XY in E and F, respectively. Show that $ar(\Delta ABE) = ar(\Delta ACF)$.



7. Diagonals AC and BD of a quadrilateral $ABCD$ intersect at O, such that $OB = OD$. If $AB = CD$, then show that:
   (i) ar ΔDOC = ar ΔAOB
   (ii) ar ΔDCB = ar ΔACB
   (iii) DA parallel to CB or $ABCD$ is a parallelogram

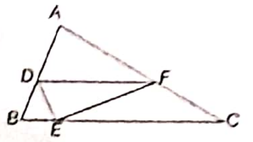


8. In the figure, $ABCD$ is a parallelogram and $PBQR$ is a rectangle. If $AP : PB = 1 : 2 = PD : DR$, what is the ratio of the area of $ABCD$ to the area of $PBQR$?



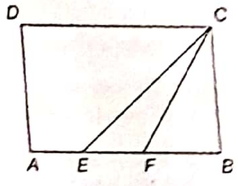
9. In the figure $AD = \frac{1}{2} DB$, $BE = EC$ and $CF = \frac{1}{3} AF$.

   If the area of $\Delta ABC = 120\ cm^2$, the area (in cm^2) of ΔDEF is



   (A) 21 (B) 35
   (C) 40 (D) 45

10. In the below diagram, $ABCD$ is a rectangle with $AE = EF = FB$. What is the ratio of the areas of ΔCEF and that of the rectangle?



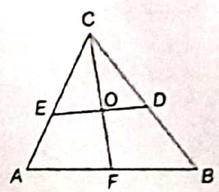
   (A) $\frac{1}{6}$ (B) $\frac{1}{8}$
   (C) $\frac{1}{9}$ (D) None of these

11. Prove that the altitudes of a triangle are concurrent.

12. In triangle ABC, point D divides side AC so that $AD : DC = 1 : 2$. Let E be the mid point of BD and F be the point of intersection of the BC and line AE. Given that the area of ΔABC is 360, what is the area of ΔEBF.

13. In triangle ABC, D, E, F are on BC, CA and AB respectively. So that $AE = AF = CD = 2$, $BD = CE = 3$, and $BF = 5$. If DE and CF intersect at O.

    Compute $\frac{OD}{OE}$



14. In ΔABC, X is a point on AB such that $AX = 2BX = 4$. If $AC = 7$ and $BC = 5$. Compute CX.

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Practice Exercise

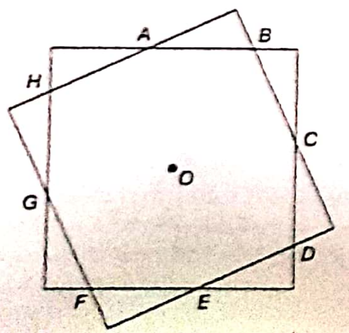
EXERCISE – I

Part I—Relevant AIME Problems

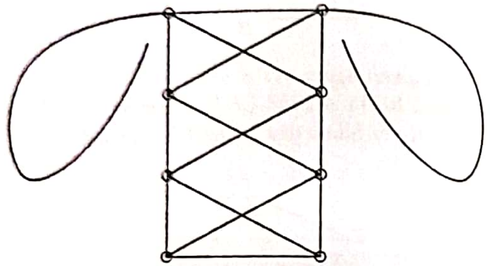
1. Rectangle $ABCD$ has sides $\overline{AB}$ of length 4 and $\overline{CB}$ of length 3. Divide $\overline{AB}$ into 168 congruent segments with points $A = P_0, P_1, \ldots, P_{168} = B$, and divide $\overline{CB}$ into 168 congruent segments with points $C = Q_0, Q_1, \ldots, Q_{168} = B$. For $1 \le k \le 167$, draw the segments $\overline{P_kQ_k}$. Repeat this construction on the sides $\overline{AD}$ and $\overline{CD}$, and then draw the diagonal $\overline{AC}$. Find the sum of the lengths of the 335 parallel segments drawn.

2. Square S_1 is 1×1. For $i \ge 1$, the lengths of the sides of square S_{i+1} are half the lengths of the sides of square S_i, two adjacent sides of square S_i are perpendicular bisectors of two adjacent sides of square S_{i+1}, and the other two sides of square S_{i+1} are the perpendicular bisectors of two adjacent sides of square S_{i+2}. The total area enclosed by at least one of S_1, S_2, S_3, S_4, S_5 can be written in the form m/n where m and n are relatively prime positive integers. Find $m - n$.

3. The two squares shown share the same center O and have sides of length 1. The length of $\overline{AB}$ is 43/99 and the area of octagon $ABCDEFGH$ is m/n where m and n are relatively prime positive integers. Find $m + n$.



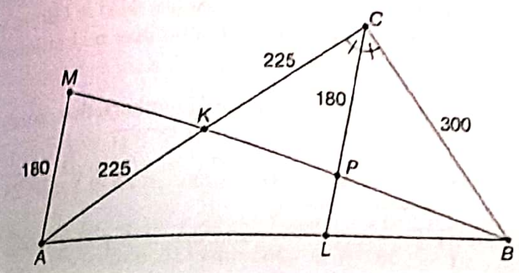
4. The 8 eyelets for the lace of a sneaker all lie on a rectangle, four equally spaced on each of the longer sides. The rectangle has a width of 50 mm and a length of 80 mm. There is one eyelet at each vertex of the rectangle. The lace itself must pass between the vertex eyelets along a width side of the rectangle and then crisscross between successive eyelets until it reaches the two eyelets at the other width side of the rectangle as shown. After passing through these final eyelets, each of the ends of the lace must extend at least 200 mm farther to allow a knot to be tied. Find the minimum length of the lace in millimeters.



5. The diagram shows a rectangle that has been dissected into nine non-overlapping squares. Given that the width and the height of the rectangle are relatively prime positive integers, find the perimeter of the rectangle.

6. One base of a trapezoid is 100 units longer than the other base. The segment that joins the midpoints of the legs divides the trapezoid into two regions whose areas are in the ratio 2 : 3. Let x be the length of the segment joining the legs of the trapezoid that is parallel to the bases and that divides the trapezoid into two regions of equal area. Find the greatest integer that does not exceed $x^2/100$.

7. Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, find LP.



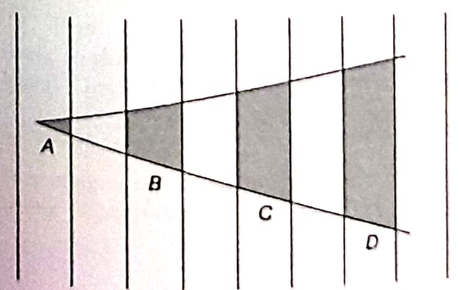
8. In triangle ABC, $AB = 125$, $AC = 117$ and $BC = 120$. The angle bisector of angle A intersects $\overline{BC}$ at point L, and the angle bisector of angle B intersects $\overline{AC}$ at point K. Let M and N be the feet of the perpendiculars from C to $\overline{BK}$ and $\overline{AL}$, respectively. Find MN.

9. Point B is on $\overline{AC}$ with $AB = 9$ and $BC = 21$. Point D is not on $\overline{AC}$ so that $AD = CD$, and AD and BD are integers. Let s be the sum of all possible perimeters of ΔACD. Find s.

10. Square $ABCD$ has sides of length 1. Points E and F are on $\overline{BC}$ and $\overline{CD}$ respectively, so that ΔAEF is equilateral. A square with vertex B has sides that are parallel to those of $ABCD$ and a vertex on $\overline{AE}$. The length of a side of this smaller square is $\frac{a - \sqrt{b}}{c}$ where a, b and c are positive integers and b is not divisible by the square of any prime. Find $a + b + c$.

11. In trapezoid $ABCD$, leg $\overline{BC}$ is perpendicular to bases $\overline{AB}$ and $\overline{CD}$, and diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular. Given that $AB = \sqrt{11}$ and $AD = \sqrt{1001}$, find BC^2.

12. An angle is drawn on a set of equally spaced parallel lines as shown. The ratio of the area of shaded region C to the area of shaded region B is 11/5. Find the ratio of shaded region D to the area of shaded region A.



13. In triangle ABC, $AB = 13$, $BC = 15$ and $CA = 17$. Point D is on $\overline{AB}$, E is on $\overline{BC}$, and F is on $\overline{CA}$. Let $AD = p \cdot AB$, $BE = q \cdot BC$, and $CF = r \cdot CA$, where p, q, and r are positive and satisfy $p + q + r = 2/3$ and $p^2 + q^2 + r^2 = 2/5$. The ratio of the area of triangle DEF to the area of triangle ABC can be written in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

14. Rhombus $PQRS$ is inscribed in rectangle $ABCD$ so that vertices P, Q, R, and S are interior points on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. It is given that $PB = 15$, $BQ = 20$, $PR = 30$, and $QS = 40$. Let $\frac{m}{n}$, in lowest terms, denote the perimeter of $ABCD$. Find $m + n$.

15. Triangle ABC is isosceles with $AC = BC$ and $\angle ACB = 106°$. Point M is in the interior of the triangle so that $\angle MAC = 7°$ and $\angle MCA = 23°$. Find the number of degrees in $\angle CMB$.

16. Let ABC be equilateral, and D, E and F be the midpoints of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively. There exist points P, Q and R on $\overline{DE}$, $\overline{EF}$ and $\overline{FD}$ respectively, with the property that P is on $\overline{CQ}$, Q is on $\overline{AR}$, and R is on $\overline{BP}$. The ratio of the area of triangle ABC to the area of triangle PQR is $a + b\sqrt{c}$ where a, b and c are integers, and c is not divisible by the square of any prime. What is $a^2 + b^2 + c^2$?

17. Triangle ABC is a right triangle with $AC = 7$, $BC = 24$, and right angle at C. Point M is the midpoint of AB, and D is on the same side of line AB as C so that $AD = BD = 15$. Given that the area of triangle CDM may be expressed as $\frac{m\sqrt{n}}{p}$, where m, n and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime, find $m + n + p$.

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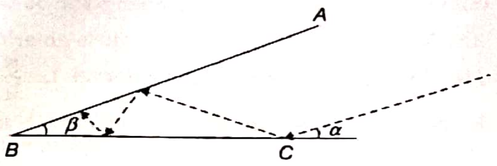
18. A beam of light strikes $\overline{BC}$ at point C with angle of incidence $\alpha = 19.94°$ and reflects with an equal angle of reflection as shown. The light beam continues its path, reflecting off line segments $\overline{AB}$ and $\overline{BC}$ according to the rule: angle of incidence equals angle of reflection. Given that $\beta = \alpha/10 = 1.994°$ and $AB = BC$, determine the number of times the light beam will bounce off the two line segments. Include the first reflection at C in your count.



19. In triangle ABC, $AB = \sqrt{30}$, $AC = \sqrt{6}$, and $BC = \sqrt{15}$. There is a point D for which $\overline{AD}$ bisects $\overline{BC}$, and $\angle ADB$ is a right angle. The ratio $\frac{[ADB]}{[ABC]}$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

20. In quadrilateral $ABCD$, $\angle BAD \cong \angle ADC$ and $\angle ABD \cong \angle BCD$, $AB = 8$, $BD = 10$, and $BC = 6$. The length CD may be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

21. In triangle ABC point D is on $\overline{BC}$ with $CD = 2$ and $DB = 5$ point E is on $\overline{AC}$ with $CE = 1$ and $EA = 3$, $AB = 8$ and $\overline{AD}$ and $\overline{BE}$ intersect at P. Points Q and R lie on $\overline{AB}$ so that $\overline{PQ}$ is parallel to $\overline{CA}$ and $\overline{PR}$ is parallel to $\overline{CB}$ It is given that the ratio of the area of triangle PQR to the area of triangle ABC is m/n, where m and n are relatively prime positive integers. Find $m + n$.

22. In ΔRED, $\angle DRE = 75°$ and $\angle RED = 45°$. $RD = 1$. Let M be the midpoint of segment $\overline{RD}$. Point C lies on side $\overline{ED}$ such that $\overline{RC} \perp \overline{EM}$. Extend segment $\overline{DE}$ through E to point A such that $CA = AR$. Then $AE = \frac{a - \sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer. Find $a + b + c$.

23. In triangle ABC, it is given that angles B and C are congruent. Points P and Q lie on $\overline{AC}$ and $\overline{AB}$ respectively, so that $AP = PQ = QB = BC$. Angle ACB is r times as large as angle APQ, where r is a positive real number. Find $\lfloor 1000r \rfloor$.

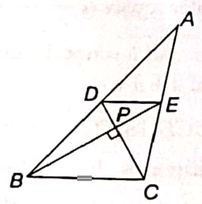
24. Given that x, y and z are real numbers that satisfy $x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}$, $y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}$, $z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}}$ and that $x + y + z = \frac{m}{\sqrt{n}}$, where m and n are positive integers and n is not divisible by the square of any prime, find $m + n$.

25. On square $ABCD$, point E lies on side AD and point F lies on side BC, so that $BE = EF = FD = 30$. Find the area of the square $ABCD$.

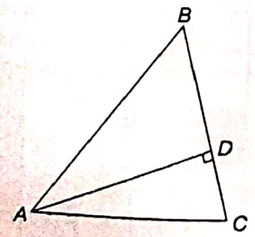
Part II—Additional IOQM/PRMO Level Problems

1. When extending AB, BC, CA of ΔABC to B', C', A', respectively, such that $AB' = 2AB$, $CC' = 2BC$, $AA' = 3CA$. If area of ΔABC is 1, find the area of $\Delta A'B'C$.

2. In ΔABC, D and E are the mid-points of the sides AB and AC respectively, CD and BE intersect at P with $\angle BPC = 90°$. Suppose $BD = 8\sqrt{5}$ and CE = $6\sqrt{5}$. Find BC.



3. In ΔABC, $\angle BAC = 45°$, D is a point on BC such that AD is perpendicular to BC. If $BD = 3$ cm and $DC = 2$ cm, and the area of the ΔABC is x cm^2. Find the value of x.

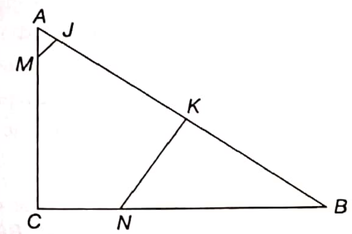


4. Let ABC be any triangle. Let D and E be the points respectively in the segments of AB and BC such that $AD = 7DB$ and $BE = 10EC$. Assume that AE and CD meet at point F. Determine $[k]$, where k is the real number such that $AF = k \times FE$.

5. In an acute-angled triangle ABC, points D, E, and F are the feet of the perpendiculars from A, B, and C onto BC, AC, and AB, respectively. Suppose $\sin A = \frac{3}{5}$ and $BC = 39$. Find the length of AH, where H is the intersection AD with BE.

6. Let O be an interior point of ΔABC. Extend AO to meet the side BC at D. Similarly, extend BO and CO to meet CA and AB, respectively at E and F. If $AO = 30$, $FO = 20$, $BO = 60$, $DO = 10$, and $CO = 20$, find EO.

7. In ΔABC, $AB = 13$, $AC = 5$, and $BC = 12$. Points M and N lie on $\overline{AC}$ and $\overline{BC}$, respectively, with CM = CN = 4. Points J and K are on AB so that $\overline{MJ}$ and $\overline{NK}$ are perpendicular to $\overline{AB}$. What is the whole part of area of pentagon $CMJKN$?



8. In triangle ABC, side AC and the perpendicular bisector of BC meet in point D, and BD bisects $\angle ABC$. If $AD = 9$ and $DC = 7$, the area of triangle $ABD = a\sqrt{5}$ then the value of a?

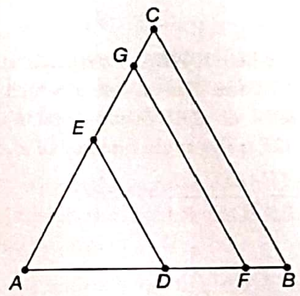
9. In ΔABC, we have $AB = 1$ and $AC = 2$. Side BC and the median from A to BC have the same length. If the length of BC is $\sqrt{a}$ then the value of a?

10. Triangle ABC is an isosceles right triangle with $AB = AC = 3$. Let M be the mid-point of hypotenuse $\overline{BC}$. Points I and E lie on sides $\overline{AC}$ and AB, respectively, so that $AI > AE$ and $AIME$ is a cyclic quadrilateral. Given that triangle EMI has area 2, the length CI can be written as $\frac{a - \sqrt{b}}{c}$, where a, b and c are positive integers and b is not divisible by the square of any prime. What is the value of $a + b + c$?

11. In ΔPAT, $\angle P = 36°$, $\angle A = 56°$, and $PA = 10$. Points U and G lie on sides TP and TA, respectively, so $PU = \overline{AG} = 1$. Let M and N be the mid-points of segments $\overline{PA}$ and $\overline{UG}$, respectively. What is the degree measure of the acute angle formed by lines MN and PA?

12. Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3AB$. Similarly extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3\,BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3CA$. The ratio of the area of $\Delta A'B'C'$ to the area of ΔABC is $Q : 1$. What is Q?

13. Triangle ABC is equilateral with $AB = 1$. Points E and G are on $\overline{AC}$ and points D and F are on $\overline{AB}$ such that both $\overline{DE}$ and $\overline{FG}$ are parallel to $\overline{BC}$. Furthermore, triangle ADE and trapezoids $DFGE$ and $FBCG$ all have the same perimeter. $DE + FG$ can be written as $\frac{r}{s}$ where r and s are co-prime. What is $r + s$?



14. The angles in a particular triangle are in arithmetic progression, and the side lengths are 4, 5, x. The sum of the possible values of x equals $a + \sqrt{b} + \sqrt{c}$ where a, b, and c are positive integers. What is $a + b + c$?

15. In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$ and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

16. Points B and C lie on $\overline{AD}$. The length of $\overline{AB}$ is 4 times the length of $\overline{BD}$, and the length of $\overline{AC}$ is 9 times the length of $\overline{CD}$. The length of $\overline{BC}$ × S = length of $\overline{AD}$. What is S?

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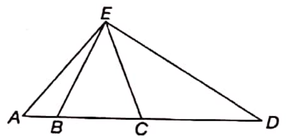
17. Point P is inside equilateral ΔABC. Points Q, R, and S are the feet of the perpendiculars from P to $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$, respectively. Given that $PQ = 1$, $PR = 2$, and $PS = 3$, what is AB^2?

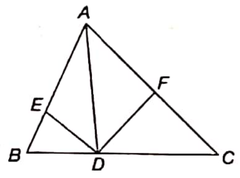
18. In triangle ABC, $AB = (b^2 - 1)$ cm, $BC = a^2$ cm and $AC = 2a$ cm, where a and b are positive integers greater than 1. Find the value of $a - b$.

19. In the figure below, ADE is a triangle with $\angle AED = 120°$, and B and C are points on side AD such that BCE is an equilateral triangle. If $AB = 4$ cm, $CD = 16$ cm and $BC = x$ cm, find the value of x.

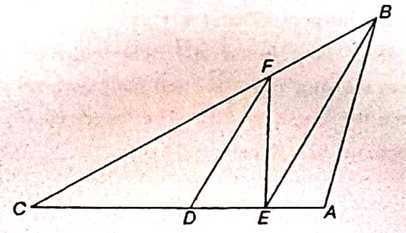


20. In the triangle ABC, $AB = 8$, $BC = 7$, and $CA = 6$. Let E be the point on BC such that $\angle BAE = 3\angle EAC$. Find $\frac{4AE^2}{5}$.

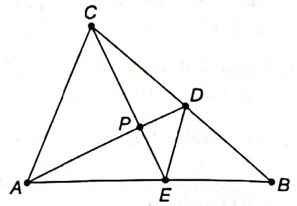
21. In the figure below, ABC is a triangle and D is a point on side BC, Point E is on side AB such that DE is the angle bisector of $\angle ADB$ and point F is on side AC such that DF is the angle bisector of $\angle ADC$. Find the value of $\frac{AE}{EB} \cdot \frac{BD}{DC} \cdot \frac{CF}{FA}$.



22. In the figure below, ABC is a triangle with $AB = 10$ cm and $BC = 40$ cm. Points D and E lie on side AC and point F on side BC such that EF is parallel to AB and DF is parallel to EB. Given that BE is an angle bisector of $\angle ABC$ and that $AD = 13.5$ cm, find the length of CD in cm.



23. In ΔABC, medians $\overline{AD}$ and $\overline{CE}$ intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $2[AEDC]$?



24. A flagpole is originally 5 meters tall. A hurricane snaps the flagpole at a point x meters above the ground so that the upper part, still attached to the stump, touches the ground 1 meter away from the base. What is $[x]$ where [:] is the greatest integer function?

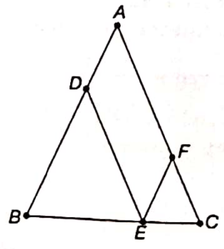
25. Distinct points A, B, C, and D lie on a line, with $AB = BC = CD = 1$. Points E and F lie on a second line, parallel to the first, with $EF = 1$. A triangle with positive area has three of the six points as its vertices. How many possible values are there for the area of the triangle?

26. Two equilateral triangles are contained in a square whose side length is $2\sqrt{3}$. The bases of these triangles are the opposite sides of the square and their intersection is a rhombus. If the area of the rhombus is $8\sqrt{k} - 12$, find k.

27. A triangle ΔABC is inscribed in a circle of radius 1, with $\angle BAC = 60°$. Altitudes AD and BE of ΔABC intersect at H. Find the smallest possible value of the length of the segment AH.

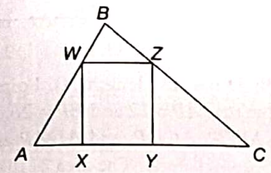
28. Two sides of a triangle have lengths 10 and 15. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?

29. In ΔABC, $AB = AC = 28$ and $BC = 20$. Points D, E, and F are on sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ respectively, such that $\overline{DE}$ and $\overline{EF}$ are parallel to $\overline{AC}$ and $\overline{AB}$, respectively. What is the perimeter of parallelogram ADEF?



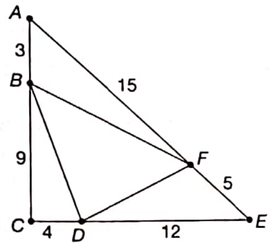
30. In ΔABC, $\angle C = 90°$ and $AB = 12$. Squares $ABXY$ and $ACWZ$ are constructed outside of the triangle. The points X, Y, Z, and W lie on a circle. The perimeter of the triangle can be written as $a(1 + \sqrt{b})$. Find the product of a and b.

31. Right ΔABC has $AB = 3$, $BC = 4$, and $AC = 5$. Square $XYZW$ is inscribed in ΔABC with X and Y on $\overline{AC}$, W on $\overline{AB}$, and Z on $\overline{BC}$. What is the side length of the square to the nearest integer?

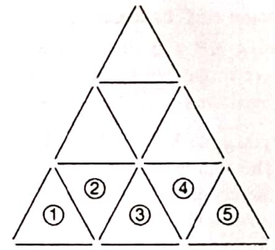


32. In a triangle with integer side lengths, one side is three times as long as the second side, and the length of the third side is 15. What is the greatest possible perimeter of the triangle?

33. In right triangle ΔACE, we have $AC = 12$, $CE = 16$, and $EA = 20$. Points B, D, and F are located on $\overline{AC}$, $\overline{CE}$, and $\overline{EA}$, respectively, so that $AB = 3$, $CD = 4$, and $EF = 5$. What is the difference of the area of ΔACE to that of ΔDBF?



34. A large equilateral triangle is constructed by using toothpicks to create rows of small equilateral triangles. For example, in the figure we have 3 rows of small congruent equilateral triangles, with 5 small triangles in the base row. Total N number of toothpicks would be needed to construct a large equilateral triangle if the base row of the triangle consists of 2003 small equilateral triangles. Then find the sum of all digits of N.



35. Points A, B, C, D, E, and F lie, in that order, on $\overline{AF}$, dividing it into five segments, each of length 1. Point G is not on line AF. Point H lies on $\overline{GD}$, and point J lies on $\overline{GF}$. The line segments $\overline{HC}$, $\overline{JE}$, and $\overline{AG}$ are parallel. Find $\left(\frac{3HC}{JE}\right)$.

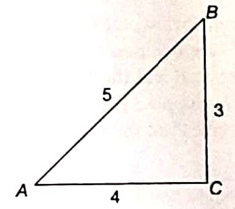
36. In triangle ABC, $\angle ABC = 120°$, $\overline{AB} = 3$ and $BC = 4$. If perpendiculars constructed to $\overline{AB}$ at A and to $\overline{BC}$ at C meet at D, then $\sqrt{3}$ CD equals?

37. Two congruent $30° - 60° - 90°$ triangles are placed so that they overlap partly and their hypotenuses coincide. If the hypotenuse of each triangle is 12, and the area common to both triangles is given by A. Find the value of $\sqrt{3}$ A.

38. In ΔABC with right angle at C, altitude CH and median CM trisects the right angle. If the area of ΔCHM is 10, then the area of ΔABC is

39. In a triangle with side of lengths a, b, and c, $(a + b + c)(a + b - c) = 3ab$. The measure of the angle opposite to the side of length c is?

40. A paper triangle with sides of lengths 3, 4, and 5 inches, as shown, is folded so that point A falls on point B. Then the length in inches of the crease is L. Find the value of $8L$.



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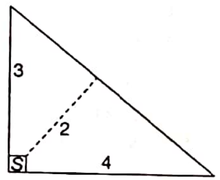
41. Right triangle ABC has leg lengths $AB = 20$ and $BC = 21$. Including $\overline{AB}$ and $\overline{BC}$, how many line segments with integer length can be drawn from vertex B to a point on hypotenuse $\overline{AC}$?

42. Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse E is 2 units. Let the fraction of the field planted can be written as $\frac{x}{y}$. Find the value of $10(y - x)$ where x, y are co-prime numbers.

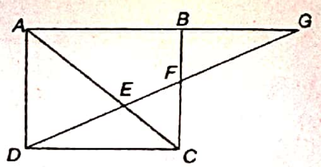


43. Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the mid-point of $\overline{AB}$, and let E be the mid-point of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

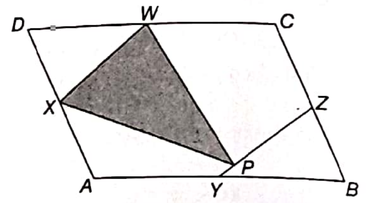
44. A square with side length x is inscribed in a right triangle with sides of length 3, 4, and 5 so that one vertex of the square coincides with the right-angled vertex of the triangle. A square with side length y is inscribed in another right-angled triangle with sides of length 3, 4, and 5 so that one side of the square lies on the hypotenuse of the triangle. Find the value of $(7x + 37y)$.

45. Rectangle $ABCD$ has $AB = 3$ and $BC = 4$. Point E is the foot of the perpendicular from B to diagonal $\overline{AC}$. If the area of ΔADE is given by P. Find $25P$.

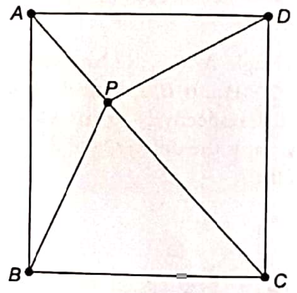
46. In the diagram below, $ABCD$ is a square. The points A, B, and G are collinear. The line segments AC and DG intersect at E and the line segments DG and BC intersect at F. Suppose that $DE = 15$ cm, $EF = 9$ cm, and $FG = x$ cm. Find the value of x.



47. W, X, Y, and Z are the mid-points of the four sides of parallelogram $ABCD$. P is a point on the line segment YZ. What percent of the area of parallelogram $ABCD$ is triangle PXW?

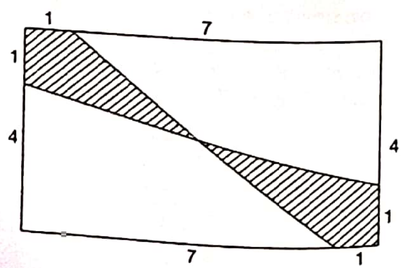


48. Square $ABCD$ has side length 30. Point P lies inside the square so that $AP = 12$ and $BP = 26$. The centroids of ΔABP, ΔBCP, ΔCDP, and ΔDAP are the vertices of a convex quadrilateral. The area of that quadrilateral is A. What is $\frac{A}{4}$?



49. The ratio of the short side of a certain rectangle to the long side is equal to the ratio of the long side to the diagonal. The square of the ratio of the short side to the long side of this rectangle is in the form of $\frac{\sqrt{a} - b}{c}$. What is $(a + b - c)^2$?

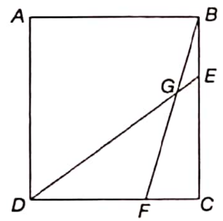
50. The area of the shaded region of the given 8×5 rectangle is an improper fraction with co-prime digits. What is the sum of numerator and denominator?



51. For some positive integers p, there is a quadrilateral $ABCD$ with positive integer side lengths, perimeter p, right angles at B and C, $AB = 2$, and $CD = AD$. How many different values of $p < 2015$ are possible?

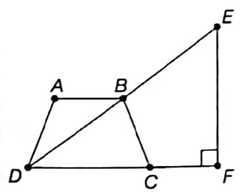
52. Trapezoid $ABCD$ has $AD \parallel BC$, $BD = 1$, $\angle DBA = 23°$, and $\angle BDC = 46°$. The ratio $BC : AD$ is $9 : 5$. $CD = \frac{p}{q}$ where $gcd\ (p, q) = 1$. What is $p^2 + q^2$?

53. Given that $ABCD$ is a square. Points E and F lie on the side BC and CD, respectively, such that $BE = CF = \frac{1}{3}AB$. G is the intersection of BF and DE. If $\frac{Area\ of\ ABGD}{Area\ of\ ABCD} = \frac{m}{n}$ is in the lowest term, find the value of $m + n$.

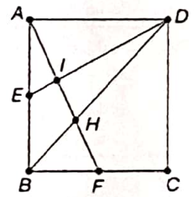


54. A regular octagon $ABCDEFGH$ has sides of length two. Then the area of ΔADG is given by $(a + b\sqrt{c})$. Find the product of a, b, & c.

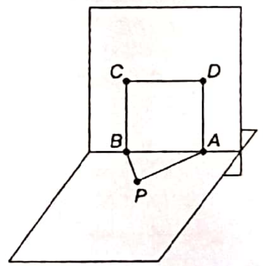
55. In the figure, $ABCD$ is an isosceles trapezoid with side lengths $AD = \overline{BC} = 5$, $AB = 4$, and $DC = 10$. The point $\overline{C}$ is on $\overline{DF}$ and B is the mid-point of hypotenuse $\overline{DE}$ in the right triangle DEF. Then $CF =$



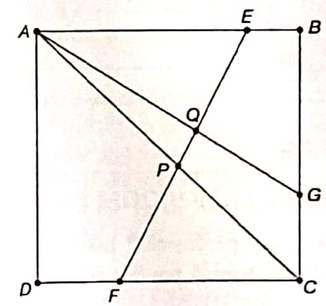
56. If $ABCD$ is a 2×2 square, E is the mid-point of $\overline{AB}$, F is the mid-point of $\overline{BC}$, $\overline{AF}$, and $\overline{DE}$ intersect at I, and BD and AF intersect at H, and the area of quadrilateral $BEIH$ is $\frac{k}{2k+1}$, find k.



57. Triangle PAB and square $ABCD$ are in perpendicular planes. Given that $PA = 3$, $PB = 4$, and $AB = 5$, what is PD^2?



58. Rectangle ABCD has AB = 5 and BC = 4. Point E lies on $\overline{AB}$ so that EB = 1, point G lies on $\overline{BC}$ so that CG = 1 and point F lies on $\overline{CD}$ so that DF = 2. Segments $\overline{AG}$ and $\overline{AC}$ intersect $\overline{EF}$ at Q and P, respectively. What is the value of $91\left(1 - \frac{PQ}{EF}\right)$?



59. Three unit squares and two line segments connecting two pairs of vertices are shown. If the area of ΔABC is N, find $5N$.

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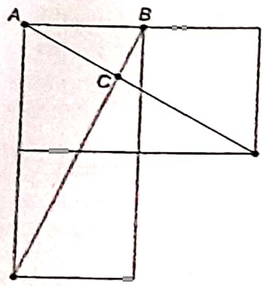
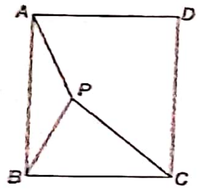
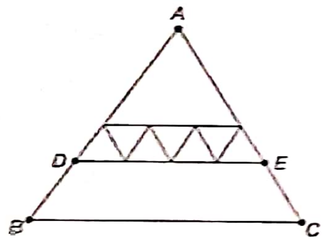


60. In the following diagram, ABCD is a square with $PA = a$, $PB = 2a$ and $PC = 3a$. Find the value of x where $x° = 180° - \angle APB$.



61. All of the triangles in the diagram below are similar to isosceles triangle ABC, in which AB = AC. Each of the 7 smallest triangles has area 1, and ΔABC has area 40. What is the area of trapezoid DBCE?



EXERCISE – II

Previous Years' PRMO/IOQM Problems

1. A triangle with perimeter 7 has integer-side length. What is the maximum possible area of such a triangle? **[PRE-RMO-2012]**

2. In ΔABC, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is the length of segment BD? **[PRE-RMO-2012]**

3. In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on line segment CD so that $DF = 1$ and $GC = 2$. Line AF and BG intersect at E. What is the area of ΔAEB? **[PRE-RMO-2012]**

4. $ABCD$ is a square and $AB = 1$. Equilateral triangle AYB and CXD are drawn such that X and Y are inside the square. What is the length of XY? **[PRE-RMO-2012]**

5. Let AD and BC be the parallel sides of a trapezium $ABCD$. Let P and Q be the mid-points of the diagonals AC and BD. If $AD = 16$ and $BC = 20$, what is the length of PQ? **[PRE-RMO-2013]**

6. Let ABC be an equilateral triangle. Let P and S be point on AB and AC, respectively, and let Q and R be points on BC such that $PQRS$ is a rectangle. If $PQ = \sqrt{3}\ PS$ and the area of $PQRS$ is $28\sqrt{3}$, what is the length of PC? **[PRE-RMO-2013]**

7. Let A_1, B_1, C_1, D_1 be the mid-points of the sides of a convex quadrilateral $ABCD$ and let A_2, B_2, C_2, D_2 be the mid-points of the sides of the quadrilateral $A_1B_1C_1D_1$. If $A_2B_2C_2D_2$ is a rectangle with sides 4 and 6, then what is the product of the lengths of the diagonals of $ABCD$? **[PRE-RMO-2013]**

8. In a triangle ABC with $\angle BCA = 90°$, the perpendicular bisector of AB intersects segments AB and AC at X and Y, respectively. If the ratio of the area of quadrilateral $BXYC$ to the area of triangle ABC is 13 : 18 and $BC = 12$ then what is the length of AC? **[PRE-RMO-2013]**

9. In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 17. What is the greatest possible perimeter of the triangle? **[PRE-RMO-2014]**

10. In a triangle ABC, X and Y are points on the segments AB and AC, respectively, such that $AX : XB = 1 : 2$ and $AY : YC = 2 : 1$. If the area of triangle AXY is 10 then what is the area of triangle ABC? **[PRE-RMO-2014]**

11. Let XOY be a triangle with $\angle XOY = 90°$. Let M and N be the mid-points of legs OX and OY, respectively. Suppose that $XN = 19$ and $YM = 22$. What is XY? **[PRE-RMO-2014]**

12. A 2×3 rectangle and a 3×4 rectangle are contained within a square without overlapping at any interior point, and the side of the square are parallel to thi sides of the two given rectangle. What is the smallest possible area of the square? **[PRE-RMO-2015]**

13. What is the greatest possible perimeter of a right-angled triangle with integer side lengths if one sides has length 12? **[PRE-RMO-2015]**

14. In a rectangle $ABCD$, E is the mid-point of AB, F is a point on AC such that BF is perpendicular to AC and FE perpendicular to BD. Suppose $BC = 8\sqrt{3}$. Find DB. **[PRE-RMO-2015]**

15. Suppose the altitudes of a triangle are 10, 12 and 15. What is the semi-perimeter? **[PRE-RMO-2015]**

16. What is the number of triples (a, b, c) of positive such that (i) $a < b < c < 10$ and (ii) a, b, c, 10 from the sides of a quadrilateral? **[PRE-RMO-2017]**

17. Let P be an interior point of a triangle ABC whose side lengths are 26, 65, 78. The line through P parallel to BC meets AB in K and AC in L. The line through P parallel to CA meets BC in M and BA in N The line through P parallel to AB meets CA in S and CB in T. If KL, MN, ST are of equal lengths, find this common length. **[PRE-RMO-2017]**

18. Let $ABCD$ be a rectangle and let E and F be points on CD and BC, respectively, such that area $(ADE) = 16$, area$(CEF) = 9$ and area$(ABF) = 25$. What is the area of triangle AEF? **[PRE-RMO-2017]**

19. Consider the areas of the four triangle obtained by drawing the diagonals AC and BD of trapezium $ABCD$. The product of these areas, takes two at a time, are computed. If among the six products so obtained, two products are 1296 and 576, determine the square roots of the maximum possible area of the trapezium of the nearest integer. **[PRE-RMO-2017]**

20. In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $(BC^2 + CA^2 + AB^2)/100$. **[PRE-RMO-2018]**

21. Let ABC be an acute-angled triangle and let H be its orthocentre. Let G_1, G_2, and G_3 be the centroids of the triangles HBC, HCA, and HAB, respectively. If the area of triangle $G_1G_2G_3$ is units, what is the area of triangle ABC? **[PRE-RMO-2018]**

22. In a triangle ABC, the median AD (with D on BC) and the angle bisector BE (with E on AC) are perpendicular to each other. If $AD = 7$ and $BE = 9$, find the integer nearest to the area of triangle ABC. **[PRE-RMO-2019]**

23. If 15 and 9 are lengths of two medians of a triangle, what is the maximum possible area of the triangle to the nearest integer? **[PRE-RMO-2019]**

24. Let ABC be an acute-angled triangle with $AB = 15$ and $BC = 8$. Let D be a point on AB such that $BD = BC$. Consider points E on AC such that $\angle DEB = \angle BEC$. If α denotes the product of all possible value of AE, find $[\alpha]$, the integer part of α. **[PRE-RMO-2019]**

25. Let $ABCD$ be a trapezium in which $AB \parallel CD$ and $AB = 3CD$. Let E be then midpoint of the diagonal BD. If $[ABCD] = n \times [CDE]$, what is the value of n? (Here $[\tau]$ denotes the area of the geometrical figure τ). **[IOQM – 2021]**

26. Let $ABCD$ be a rectangle in which $AB + BC + CD = 20$ and $AE = 9$ where E is the mid-point of the side BC. Find the area of the rectangle. **[IOQM – 2021]**

27. Let ABC be a triangle with $AB = AC$. Let D be a point on the segment BC such that $BD = 48\frac{1}{61}$ and $DC = 61$. Let E be a point on AD such that CE is perpendicular to AD and $DE = 11$. Find AE. **[IOQM – 2021]**

28. Let ABC be a triangle with $AB = 5$, $AC = 4$, $BC = 6$. The internal angle bisector of C intersects the side AB at D. Points M and N are taken on sides BC and AC, respectively, such that $DM \parallel AC$ and $DN \parallel BC$. If $(MN)^2 = \frac{p}{q}$ where p and q are relatively prime positive integers then what is the sum of the digits of $|p - q|$? **[IOQM – 2021]**

29. Let $ABCD$ be a parallelogram. Let E and F be mid-points of AB and BC respectively. The lines EC and FD intersect in P and form four triangles APB, BPC, CPD and DPA. If the area of the parallelogram is 100 sq. units, what is the maximum area in sq. units of a triangle among these four triangles? **[IOQM – 2021]**

30. In triangle ABC, let P and R be the feet of the perpendiculars from A onto the external and internal bisectors of $\angle ABC$, respectively; and let Q and S be the feet of the perpendiculars from A onto the internal and external bisectors of $\angle ACB$, respectively. If $PQ = 7$, $QR = 6$ and $RS = 8$, what is the area of triangle ABC? **[IOQM – 2021]**

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31. Three parallel lines L_1, L_2, L_3 are drawn in the plane such that the perpendicular distance between L_1 and L_2 is 3 and the perpendicular distance between L_2 and L_3 is also 3. A square $ABCD$ is constructed such that A lies on L_1, B lies on L_3 and C lies on L_2. Find the area of the square. **[IOQM – 2022]**

32. Consider the set T of all triangles whose sides are distinct prime numbers which are also in arithmetic progression. Let $\Delta \in T$ be the triangle with the least perimeter. If $a°$ is the largest angle of Δ and if L is its perimeter, determine the value of $\frac{a}{L}$.

[IOQM – 2022]

33. In parallelogram $ABCD$ the longer side is twice the shorter side. Let $XYZW$ be the quadrilateral formed by the internal bisectors of the angles of $ABCD$. If the area of $XYZW$ is 10, find the area of $ABCD$.

[IOQM – 2022]

34. In a triangle ABC, let E be the midpoint of AC and F be the midpoint of AB. The medians BE and CF intersect at G. Let Y and Z be the midpoints of BE and CF respectively. If the area of triangle ABC is 480, find the area of triangle GYZ. **[IOQM – 2023]**

35. Let ABC be a trianlge in the xy plane, where B is at the origin $(0, 0)$. Let BC be produced to D such that $BC : CD = 1 : 1$. CA be produced to E such that $CA : AE = 1 : 2$ and AB be produced to F such that $AB : BF = 1 : 3$. Let $G(32, 34)$ be the centroid of the triangle ABC and K be the centroid of the triangle DEF. Find the length GK. **[IOQM – 2023]**

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ANSWER KEYS

Know Your Progress - 1

1. Euclid II[nd] axiom, a = 40
3. 126° 4. 54° 5. (i) infinite (ii) one
7. 79°12′ 8. $\angle COD = 80°$ 9. 260°

Know Your Progress - 2

1. 20 2. $x = 45°$, $y = 26°$ 6. two ways

Know Your Progress - 3

1. 30 3. $PN = 15$ cm, $RM = 10.67$ cm 7. C 8. A 9. C 10. B 11. A 12. C
13. C 14. B 15. C

Know Your Progress - 4

3. 5 cm 8. D 9. D 10. B

Know Your Progress - 5

8. 1 : 2 9. B 10. B 12. 30 13. $\frac{5}{3}$ 14. 5

Practice Exercise

EXERCISE – I

PART I – Relevant AIME Problems

1. 840	2. 255	3. 185	4. 790	5. 260	6. 181	7. 072	8. 056	9. 380	10. 12
11. 110	12. 408	13. 061	14. 677	15. 83°	16. 083	17. 578	18. 071	19. 065	20. 069
21. 901	22. 056	23. 571	24. 009	25. 810					

PART II – Additional IOQM/PRMO Level Problems

1. 8	2. 20	3. 15	4. 77	5. 52	6. 20	7. 18	8. 14	9. 2	10. 12
11. 80	12. 37	13. 34	14. 36	15. 21	16. 10	17. 48	18. 0	19. 8	20. 27
21. 1	22. 24	23. 27	24. 2	25. 3	26. 3	27. 1	28. 12	29. 56	30. 24
31. 2	32. 43	33. 54	34. 27	35. 5	36. 10	37. 36	38. 40	39. 60	40. 15
41. 13	42. 20	43. 75	44. 72	45. 54	46. 16	47. 25	48. 50	49. 16	50. 15
51. 31	52. 41	53. 23	54. 24	55. 4	56. 7	57. 34	58. 81	59. 1	60. 45
61. 24									

EXERCISE – II

Previous Years' PRMO/IOQM Problems

1. $\frac{3\sqrt{7}}{4}$ unit	2. 3	3. 12.5	5. 2	6. 14	7. 208	8. 36	9. 49	10. 45	
11. 26	12. 25	13. 84	14. 24	15. Discounted		16. 73	17. Discounted		18. 30
19. 13	20. 24	21. 63	22. 47	23. 90	24. Bonus	25. 08	26. 19	27. 25	28. 02
29. 40	30. 84	31. 45	32. 8	33. 40	34. 10	35. 40			

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